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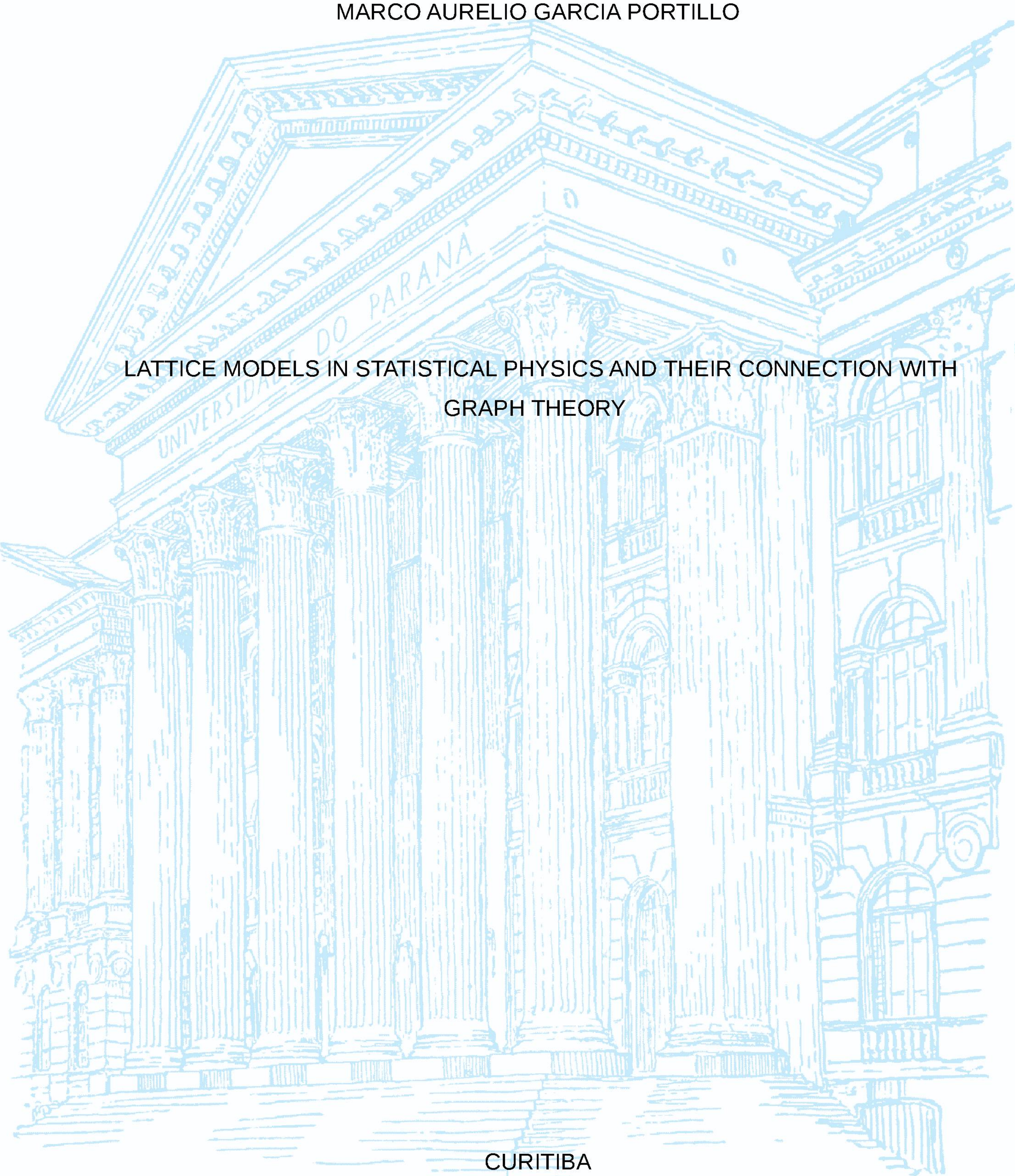

LATTICE MODELS IN STATISTICAL PHYSICS AND THEIR CONNECTION WITH GRAPH THEORY

CURITIBA

2021

MARCO AURELIO GARCIA PORTILLO

LATTICE MODELS IN STATISTICAL PHYSICS AND THEIR CONNECTION WITH GRAPH THEORY

Tese apresentada ao curso de Pós-Graduação em Física, Setor de Ciências Exatas da Universidade Federal do Paraná, como requisito parcial à obtenção do título de Doutor em Física.

Orientador: Prof. Dr. Marcos Gomes Eleutério da Luz

CURITIBA

2021

Catalogação na Fonte: Sistema de Bibliotecas, UFPR
Biblioteca de Ciência e Tecnologia

P852l Portillo, Marco Aurelio García
Lattice models in statistical physics and their connection with graph theory [recurso eletrônico] / Marco Aurelio García Portillo. – Curitiba, 2021.

Tese - Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Física, 2021.

Orientador: Marcos Gomes Eleutério da Luz.

1. Física estatística. 2. Ising, Modelo de. I. Universidade Federal do Paraná. II. Luz, Marcos Gomes Eleutério da. III. Título.

CDD: 530.15

Bibliotecária: Vanusa Maciel CRB- 9/1928

TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **MARCO AURELIO GARCÍA PORTILLO** intitulada: **"Lattice models in statistical physics and their connection with graph theory"** , sob orientação do Prof. Dr. MARCOS GOMES ELEUTÉRIO DA LUZ, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

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ACKNOWLEDGEMENTS

I thank my supervisor Marcos G. E. da Luz for his advising and all the helpful suggestions and discussions during the entire period of this work. I also thank the OAS-GCUB 2016 Partnerships Program for Education and Training (PAEC) for awarding me a doctoral scholarship (OEA/GCUB/CAPES). A special thanks also goes to my P.h.D colleagues for very helpful and friendly conversations.

I am finally grateful to my family and friends for always helping me to reach my goals and to the UFPR staff, specially to Leandro Camilo Florentino. All of you made my student life easier.

RESUMO

É bem sabido que existe uma conexão interessante entre a função geradora de spanning tree $STGF$ (spanning tree generating function) $T(z)$ e a função de partição do modelo de Ising isotrópico (sem campo magnético) na rede quadrada. Esses dois objetos são aparentemente diferentes. Por um lado, o $STGF$ $T(z)$ é definido por meio de uma equação diferencial envolvendo a função geradora de probabilidade do passeio aleatório simples numa rede vértice-transitiva e fornece a constante de spanning tree para $z = 1$. Por outro lado, o modelo de Ising é uma ferramenta criada para modelar ferromagetismo no contexto de física estatística. Neste trabalho mostramos que essa conexão é mais geral do que apenas no caso descrito. Para mostrar isso definimos uma $STGF$ estendida $T_e(z)$, que inclui todas as redes q-regulares e que se reduz a $T(z)$ quando a rede é vértice-transitiva. Fornecemos uma fórmula integral para $T_e(z)$ e a usamos para calcular $T(z)$ para todos as onze redes Arquimedianas, bem como para computar $T_e(z)$ para duas redes não vértice-transitivas relevantes, a rede martini e a rede (4, 8^2) covering/medial. Além disso, estabelecemos algumas conexões entre a $STGF$ $T(z)$ e a energia livre do modelo de Ising isotrópico (sem campo magnético) das onze redes Arquimedianas. Mostramos que a energia livre do modelo de Ising pode ser obtida a partir de $STGF$ por meio de um conjunto de funções auxiliares, $\phi_1(K), \ldots, \phi_{n_L}(K)$, onde n_L é um inteiro positivo que depende da topologia da rede. No caso $n_L = 1$ (redes quadrada, triângula, hexagonal, kagome e estrela), obtemos a propriedade adicional de que $\phi(K_c) = 1$, onde K_c é o ponto crítico do sistema. Também definimos a noção de uma função geradora de spanning tree com pesos $wSTGF$, generalizando a $eSTGF$, permitindo pesos nas arestas bem como sendo válida para redes não-regulares. Usando essa nova idéia, mostramos as relações entre o modelo de Ising anisotrópico (sem campo magnetico) e a $wSTGF$ na rede quadrada, triangular e hexagonal, e entre a $wSTGF$ e o modelo de Dímeros nas redes quadrada e triangular. Por último, encontramos que a energia livre de um modelo de Random Walk Loop Soup (definido a partir de passeios aleatórios fechados sobre uma rede) pode ser escrita em termos de $T_e(z)$ para qualquer rede q-regular.

Palavras-chaves: Física estatística. Função geradora de spanning tree. Modelo de Ising..

ABSTRACT

It has been known there exists an interesting connection between the spanning tree generating function $STGF$ $T(z)$ and the partition function (at zero-field) of the isotropic Ising model on the square lattice. Those two objects are seemingly different. On the one hand, the $STGF$ $T(z)$ is defined by means of a differential equation involving the probability generating function of the simple random walk on a vertex-transitive lattice and gives the spanning tree constant when evaluated at $z = 1$. On the other hand, the Ising is a simple model proposed to describe ferromagnetism under the realm of statistical physics. In this work we show this connection is more general than just in such particular case. To prove that, we define an extended spanning tree generating function $T_e(z)$, which include all the q-regular lattices and reduces to the $T(z)$, when the lattice is vertex-transitive. We provide an integral formula for $T_e(z)$ and use it to calculate the $T(z)$ for all the eleven Archimedean lattices and the $T_e(z)$ for two relevant non-vertex-transitive lattices, martini and (4,8^2) covering/medial. We establish some links between the $STGF$ $T(z)$, and the (zero-field) isotropic Ising free energy on the eleven Archimedean lattices. We then demonstrate that the Ising free energy can be derived from the $STGF$ via a set of auxiliary functions, $\phi_1(K), \ldots, \phi_{n_L}(K)$, where n_L is a positive integer that depends on the lattice topology. In the case $n_L = 1$ (square, triangle, hexagonal, kagome and star lattices) we obtain the additional property that $\phi(K_c) = 1$, where K_c is the critical point of the system. We also propose the notion of a weighted spanning tree generating function $wSTGF$, which generalizes the $eSTGF$, allowing positive weights on the edges and being well defined for non-regular lattices. Using this new idea, we establish relations between the (zero-field) anisotropic Ising model and the $wSTGF$ on the square, triangle and hexagonal lattices, and between the $wSTGF$ and the Dimer model on the square and triangle lattices. Additionally, we show that the free energy of a random walk loop soup model (defined from closed random walks over the lattice) can be written in terms of $T_e(z)$ for any q-regular lattice.

Key-words: Statistical physics. Spanning tree generating function. Ising model.

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1 INTRODUCTION

1.1 GRAPHS IN PHYSICS

Often, in order to theoretically understand a real physical system we must develop a mathematical model that captures at least some of its main features. The model should allow proper calculations, helping to solve the problem, namely, to predict the underlying dynamics and to determine the relevant quantities associated to the process under study.

Therefore, it should be clear that finding connections between different objects in Mathematics and Physics is of fundamental importance. On one hand, this might provide new approaches to comprehend and express many natural phenomena. On the other hand, it potentially can indicate alternative points of view for the interpretation (and even extension) of pure mathematical theories [2].

A useful tool, frequently used in physics with the above purpose, is a graph. A graph is an abstract object composed by two distinct collections of elements. Heuristically, pairs of elements of the first type, the vertices, may or may not be connected by the elements of the second type, the edges. We usually represent a graph in the plane (although in distinct occasions we also consider higher dimensional spaces) by a set of points (vertices) and segments (edges). As examples of concrete realizations of graphs in nature (in the 3D case), we mention bulk crystalline structures. Indeed, in this kind of matter organization — atoms regularly disposed and interacting through chemical bonds — can be characterized by a "visible"diagram in the Euclidean three-dimensional space. The atoms are represented by the mentioned points and the nearby interactions by the segments. There are many good references treating in general terms the standard theory of graph, here we mention Refs. [3] and [4].

Actually, graph theory has been applied in many scientific areas: the already mentioned crystallography [5], random walks [6], biology [7], computer science [8], and general network structures in natural and artificial (human related) phenomena [9, 10, 11], to cite just a few. But in special, the mathematical theory of graphs finds important associations with all of Physics [12]. For instance, it is an important source of active research in statistical mechanics and stochastic processes [13, 14]. Given the particularities of the sort of systems addressed in statistical physics, when studying a rather involving problem, the general concept of a graph is quite useful to simplify the analysis. Indeed, the original system may be modeled as a discrete graph. The constituents or parts become "sites"(or vertices) and their interactions can be represented by "links"(or edges): no edges exist between parts which do not influence each other.

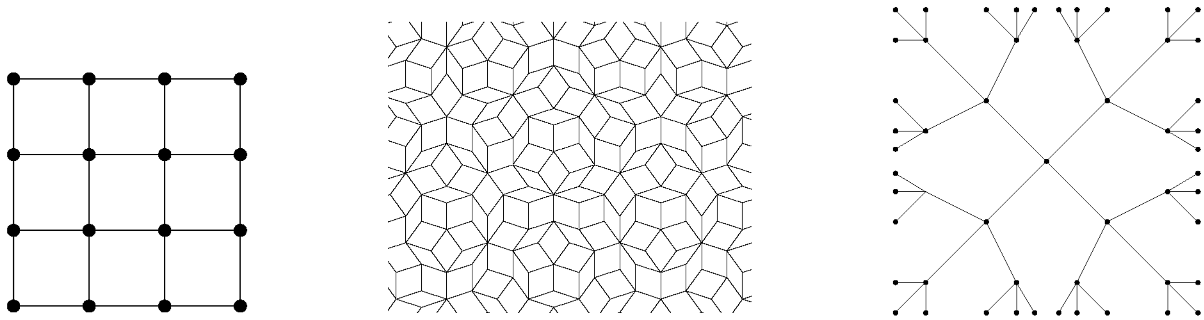
These simplified descriptions are generally called lattice models. Among the most studied (and elemental) in statistical physics are the: q-state Potts, Ising (a particular case of the Potts, but generally examined in its own right), random walks loop soups ensembles, Dimer, etc. The latter three will be discussed in details along the present work. So, only to give a "flavor"of the usefulness and construction of lattice models, we mention that the Potts have been employed to simulate different aspects of magnetism, tumor migration and social demographics [14]. For its definition, one supposes a graph G (a specific arrangement of vertices and edges). At each edge (or site) one associates a spin variable σ, which can assume q possible values (say, $1, 2, \ldots q$). Only the spins at sites which are connected to each other (e.g., vertices A and B) can interact. The local energy E_{AB} is zero if $\sigma_A \neq \sigma_B$ and a constant value otherwise. The physical attributes of the Potts model will depend on the energetic configurations resulting from the whole G structure.

There exist deep correlations between partition functions of statistical physics lattice models (like the Ising, Potts and loop soups) with graph theory [14]. Consider again the $q-$state Potts as an illustration (which is universally taken as a paradigmatic model for investigating how micro-scale nearest neighbor energy interactions in a complex system determine the macro-scale behavior of the system). Given a graph G, a notable two variables function associated to it is called Tutte polynomial (in essence, determining the degree of connectivity as well as the number of spanning trees (Chap. 2) of G) [15]. Many characteristics of the Potts model in G are closely related to the Tutte polynomial of G [14].

In this thesis, our interest is to focus on graph theory and topology (more specifically, spanning trees generating functions) and their significance to solve some statistical physics lattice models.

1.2 LATTICE MODELS AND TOPOLOGY

When formulating lattice models, many graphs or network patterns (i.e., the "medium"under which the dynamics takes place) are particularly recurrent either because their prominence in actual problems or due to adequate properties (e.g., for facilitating calculations). As example we mention the eleven Archimedean lattices (the readily recognized being the square, triangular and hexagonal ones), which are tilling of the plane by regular polygons, where every vertex is surrounded by the same sequence of polygons [1]. The Archimedean lattices are periodic, thence invariant under two independent translations in the plane. For certain lattice models, all these "nice"symmetries allow to perform explicit analytic computations, which otherwise would be unfeasible for the same systems but on other G's. The Archimedean lattices have been employed to study viruses [16] and photonic crystals [17]. Of course, there are many other important periodic lattices, as the Martini lattice (see Chap. 2), relevant in percolation theory. Also, non-periodic lattices may play a fundamental role in Physics, such as the Penrose's [18], Bethe's [19] and Cayley trees. Examples of lattices are depicted in Fig 1.



(a) (b) (c)

Figure 1 – Examples of lattices. (a) The Archimedean square (periodic). (b) Penrose tilling (non-periodic). (c) Bethe lattice (non-periodic).

Since the exact topology of a graph G — used to construct a specific lattice model — does strongly determine the physical features of the system, it is paramount to identify and to characterize the topological properties of G. A particularly important way to do so is through the idea of spanning trees. A Spanning tree on a graph G is a loopless connected subgraph that visits every vertex in G (for details see Chap. 2). The number of spanning trees $N_{ST}(G)$ of G somehow quantifies the diversity of configurations possible for G. For applications in Physics, spanning trees are related to the analysis of electric circuits [20] and of the $q-$state Pott model [21, 22].

For infinite, but periodic, lattices L there is a quantity λ_L (to be rigorously defined later on), which is usually named the spanning tree constant or the spanning tree entropy [23] (this last nomenclature, of course, indicating a close association between spanning trees and statistical physics). There is a vast literature on λ_L. For instance, it has been calculated for many lattices, including for all the Archimedean's [24, 25]. A valuable function is the spanning tree generating function STGF $T(z)$ [26], such that $T(1) = \lambda_L$.

Very recently, it has been conjectured [27] that $T(z)$ would be a fundamental mathematical tool to solve exactly lattice models. This has been concretely demonstrated for the Ising model in a square lattice and convincingly argued for the triangular lattice [27]. Such conjecture will be one of principal guides in all the developments in the present work. However, we will show that in fact $T(z)$ must be properly generalized so that the proposal in [27] can be implemented.

1.3 SOME REPRESENTATIVE LATTICE MODELS IN STATISTICAL PHYSICS

Along the main body of this thesis we will certainly discuss the necessary details about the system we shall address. Nonetheless, to better explain in the next section our main goals with the present work, next we quickly describe key elements of our lattice models.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The $q-$state Potts model is defined by assigning a spin variable σ_i to each vertex $i \in V$. The spins can take q different values, by convention chosen as $\sigma_i = 1, 2, \dots, q$. The Hamiltonian (dimensionless energy functional) of the Potts model is thus

$$H(\sigma) = -K \sum_{(i,j)\in E} \delta_{\sigma_i \sigma_j},$$

with $K = J/k_B T$ a dimensionless coupling constant (interaction energy) and δ_{xy} is the Kronecker's delta function.

The Thermodynamic information about the Potts model is encoded in the partition function

$$Z = \sum_{\sigma} \exp[H(\sigma)],$$

where the sum is over all the spin configurations $\sigma : V \to \{1, ..., q\}$.

We will analysis only the particular case of $q = 2$. This restriction of the Potts, shortly described above, is equivalent to the Ising. Historically, the Ising model was proposed by W. Lenz in 1920 and solved in 1D by his PhD student, E. Ising, in 1924-1925. We mention that the R. Potts, advised by C. Domb, developed the lattice system which bears his name in 1951 (a generalization of Ashkin-Teller system of 1943).

The 2D Ising model in the thermodynamic limit was exactly solved by Onsager in 1944 [28] (for a square lattice). Onsager proof was rewritten into a modern formalism by Kaufman considering spinors (Onsager's original paper used quaternions). Later, a novel approach was introduced by Kac and Ward in 1952 [29], subsequently refined by Feynmann [30]. The Ising model was originally introduced to understand the phase transition of magnets, from a ferromagnetic to a paramagnetic behavior. In a very simplified description, a magnet is a material whose atoms are arranged in a regular crystalline structure. Each atom carries a magnetic moment called spin. The Ising model assumes that the spins are located at the vertices of a graph $G = (V, E)$ and are restricted to point only along a single particular direction, either up (+1) or down (-1) [31]. It follows that a microstate of the system, called a spin configuration, mathematically is a function $\sigma : V \to \{-1, 1\}$ (see Fig. 1.2).

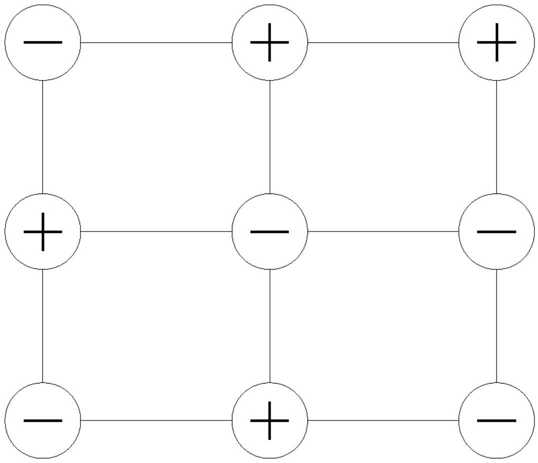


Figure 2 – A very simple example of a spin configuration $\sigma : V \to \{-1, 1\}$ in a 3×3 square lattice.

The 2D Ising model has been analytically calculated for periodic lattices such as square, triangle, hexagonal and kagome [28, 32, 33]. Moreover, it has also been studied in non-periodic networks such as the Bethe [19] and Penrose [18] lattices. To date, the exact solution of the 3D Ising model is still a major challenging in Mathematical Physics.

The Dimer model [34, 35] is another paradigmatic system in theoretical Physics. A dimer is an extremely idealized polymer with only two atoms. A dimer *covering* of a graph G is a collection of edges that covers all the vertices of G exactly once. In other words, each vertex is the endpoint of a unique edge. Even further, one can think of vertices of G as univalent atoms, each bonding to exactly one neighbor. Dimer coverings are also called perfect matchings. Mathematically, the dimer model is the study of natural measures ("Gibbs measures") on the set of dimer coverings of a graph [36] , usually a periodic planar graph such as Z^2.

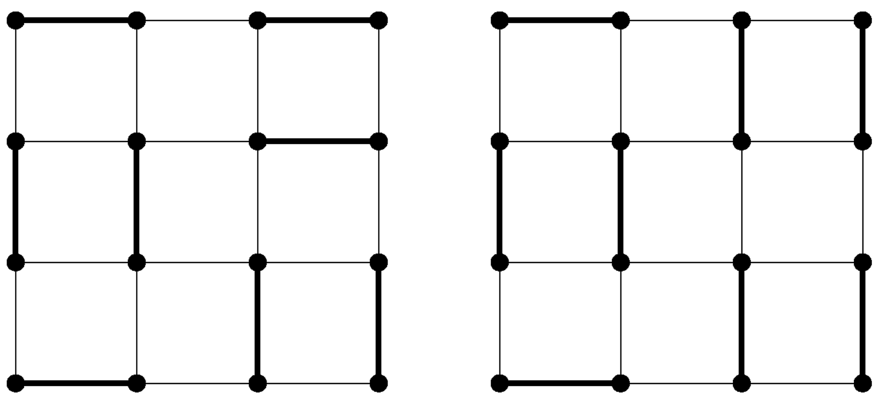


Figure 3 – Two different Dimer configuration for the Grid graph 4x4

A last lattice model which we mention — being of great interest in Mathematical

Physics — is the Random Walk Loop Soup (RWLS). It can be described as a Poisson process of lattice loops, or a lattice gas of loops since it fits within the ideal gas framework of statistical mechanics [37]. Its rigorous definition is a bit technical (presented in Chap. 2), but we can intuitively think about all the distinct loops formed with vertices and edges of G, moreover combing these loops in distinct ways to form "chains". Hence, this "soup"of closed patterns constitutes an ensemble and the aim is to study the statistical properties of these objects, associating energies to their different configurations.

The RWLS has been introduced in [38] as a proper discrete version of the Brownian loop soup. This system has been extensively studied due to its relation with discrete Gaussian free fields [39] and with conformal loop ensembles [40]. Also, a non-backtracking loop soup has been developed in [41], where it has been proved that the resulting model partition function is likewise associated with Gaussian free fields.

The specific lattice models we are going to investigate are the ones summarized above (see the following Chapters). Also, all the lattices G which we are going to consider are planar and periodic. However, we should observe that there are many other lattices system for which graph theory and topology are fundamental, for instance, topological lattice models in four dimensions [42] and Ising models on sphere-like lattices [43].

1.4 PLAN OF THE THESIS

In this thesis we are interested in finding connections between spanning trees and lattice systems in Statistical Physics, specifically, Ising, RWLS and Dimer models. The Ising model studied in this work will be with zero magnetic field ($B = 0$). We do so by analyzing (and extending) the notion of a spanning tree generating function $STGF\ T(z)$ (originally defined in [26]). It was shown in [27] that the $STGF\ T(z)$ and the partition function Z of the isotropic Ising model in square and triangular lattices are related to each other for any value of the temperature. A key issue is then if these finding can also be valid for other lattices and even for other models. The main purpose of the present work is to present an affirmative answer to this question in some particular, nevertheless noticeably, situations.

We organize the work as follows. Each Chapter (with a general structure independent of the others) is self-contained and address a specific set of problems. However, all the Chapters are interconnected through the general proposal underlying the present Thesis.

- In Chapter 2 we start reviewing the fundamental concepts of graph theory (necessary for our purposes), spanning trees, distinct symmetries of periodic graphs (among them Archimedean lattices), and the rigorous theory of random walks on graphs. Then, we present a formula for the spanning tree generating function which generalizes that in [26], called $STGF$. Our $eSTGF$ satisfies a differential equation involving the lattice Green function (LGF), but it also can be given in an integral form. Moreover, our expression leads to the spanning tree constant when evaluated at $z = 1$. For the Archimedean lattices we show that $eSTGF = STGF$. The $STGF$ for the eleven Archimedean lattices are then derived (seven of them for the very first time in the literature). Finally, the free energy of RWLS model for arbitrary regular periodic lattices is written in terms of the corresponding eSTGF. This provides a second example, besides the Ising model, in which such association is possible.

- Chapter 3 starts discussing important topological concepts of periodic lattices, useful in the solution of the Ising model. Then, we present the isotropic Ising model solution for all the eleven Archimedean lattices as well as for the Martini lattice. We lastly use the eSTGF of Chapter 2 to prove that in all the eleven cases the conjecture in [27] indeed holds true.

- In Chapter 4 we develop our final and most important extension of an expression for the spanning tree generating function. We derive an expression which maintain all the desired previous properties of $STGF$ and $eSTGF$ and that is also valid for lattices that can be non-regular and weighted. We next use such spanning tree generating function to analyze the Ising model with arbitrary couplings (anisotropic Ising model) on the square, triangle an hexagonal lattices and the Dimer model on the square and triangle lattices, obtaining connections between such models and spanning trees.

- A final Conclusion Chapter summarizes our results and put in perspective eventual future continuation for our present study.

- Very technical calculations are left to Appendices.

2 SPANNING TREE GENERATING FUNCTIONS FOR INFINITE PERIODIC GRAPHS L AND SOME CONNECTIONS WITH SIMPLE CLOSED RANDOM WALKS ON L

2.1 ABSTRACT

A spanning tree generating function $T(z)$ for d-dimensional infinite periodic graphs (or lattices) L which are also vertex-transitive has been proposed in J. Phys. A 45, 494001 (2012). The spanning tree constants λ_L of such lattices are then given by $T(z=1)$. Here a generating function $T_e(z)$, relaxing the previous condition to q-regular L's, is constructed for which also $\lambda_L = T_e(1)$. Surprisingly, the extended T_e is more amenable for explicit computations and in the vertex-transitive case leads to a new integral formula for the lattice Green function. As examples, spanning tree generating functions for all the eleven Archimedean (vertex-transitive) and two relevant non-vertex-transitive lattices, martini and the (4,8^2) covering/medial, are derived for the first time. As a further application, it is shown that the free energy of the random walk loop soup model (defined from closed random walks, the loops, over L) can be written in terms of $T_e(z)$. This demonstrates that the system critical point — existing only for $d=1$ and $d=2$ — is directly related to the spanning tree constant of L.

Keywords: Spanning tree generating functions, periodic lattices, random walk models, loop soup models

2.2 INTRODUCTION

There is a close connection between rigorous graph theory and certain areas of physics [44, 45, 46], specially those in which lattice models play a relevant role [47, 48, 49, 14]. A key point is that enumeration and topological properties are often fundamental in lattice systems [50, 23, 51, 52]. Therefore, important physical quantities — e.g., in the realm of statistical physics [30], partition function, free energy and critical temperature T_c — can be written in terms of underlying graph structures [53, 4, 6] as polynomial invariants, lattice Green functions (LGFs), spanning trees, etc [14, 54, 55, 56, 56].

An illustrative example is the relation between the spanning tree constant λ_G of an infinite periodic graph G (for proper definitions, see section 2.3) and certain discrete spin-like models on G [21, 57]. As explained in details in [27] (even with a brief historical account, refer to the refs. therein), for some problems like the Ising in the square lattice, the partition function at the critical temperature can be expressed as function of λ_{sq}.

Few universal methods can be employed to obtain the spanning tree constant. The most commonly considered are those based on the Laplacian matrix or on the Tutte polynomial of G [25, 24]. Another possibility is by means of the LGF (see, for instance, the discussion in [26]). Actually, this motivated a clever approach [26] to determine λ_G through a spanning tree generating function (STGF) $T(z)$, for which $\lambda_G = T(1)$.

The idea of generating functions in the present context is not new [58, 59, 60, 61, 62], being used to study the number of spanning trees in various situations, like for circular graphs [63]. Nonetheless, in [26] the framework has been put in very solid and general grounds, establishing $T(z)$ as an important tool for calculating λ_G. Remarkably, by analyzing the corresponding STGFs it was shown [27] that $T(z)$ and the partition function of the Ising model in square and triangular lattices are related not only at the T_c, but also for any value of the temperature.

In the original formulation [26], the STGF was defined for infinite periodic vertex-transitive graphs (or lattices) L, where $T(z)$ satisfies a differential equation involving the probability generating function (another nomenclature for the LGF) of L. Unfortunately, for many lattices of interest the necessary integration to obtain $T(z)$ might be a very hard task. Hence, the objective of this contribution is mainly twofold.

First, to broaden the STGF validity by dropping the imposition of L to be vertex-transitive (relaxing just to q–regularity, see section 2.3). We thus construct an extended spanning tree generating function (eSTGF) $T_e(z)$ for which still $\lambda_L = T_e(1)$, also leading to $T(z)$ if L is vertex-transitive. Our eSTGF is given either as a series, or it can be written as an integral (in w) with an integrand in the form $\ln[\det(1 - z\,\Lambda(w))/z^{\mathcal{S}}]$, for $\Lambda(w) \in C^{\mathcal{S}\times\mathcal{S}}$ the structure matrix of L. Such representation allows, in the vertex-transitive case, to obtain $T_e(z) = T(z)$ regardless of the LGF $P(0,z)$. For vertex-transitive lattices our protocol provides a new formula for the associated $P(0,z)$.

To illustrate the operational convenience of T_e, we determine the integral representations of $T_e(z) = T(z)$ for the eleven vertex-transitive Archimedean L's. With the exception of the square, triangular and hexagonal cases [26], as far as we know the STGF for all the other eight Archimedean lattices have not been calculated before. The LGF are also presented for such lattices. We further get $T_e(z)$ for two significant L's (specially in the study of percolation) which are non-vertex-transitives: the martini and $(4,8^2)$ covering/medial lattices. The λ_L for the former agrees with the previous known value, whereas for the latter it is accessed for the first time.

Our second goal is to provide a new example (besides that in [27]) in which the STGF, actually the eSTGF, and a lattice model are intrinsically linked. We demonstrate that the free energy of the Random Walk Loop Soup (RWLS) model [38] — valuable in different branches of theoretical physics — can be given in terms of $T_e(z)$ for arbitrary infinite periodic regular lattices L. As an application we also show that the system critical point (when existing) is directly connected with λ_L.

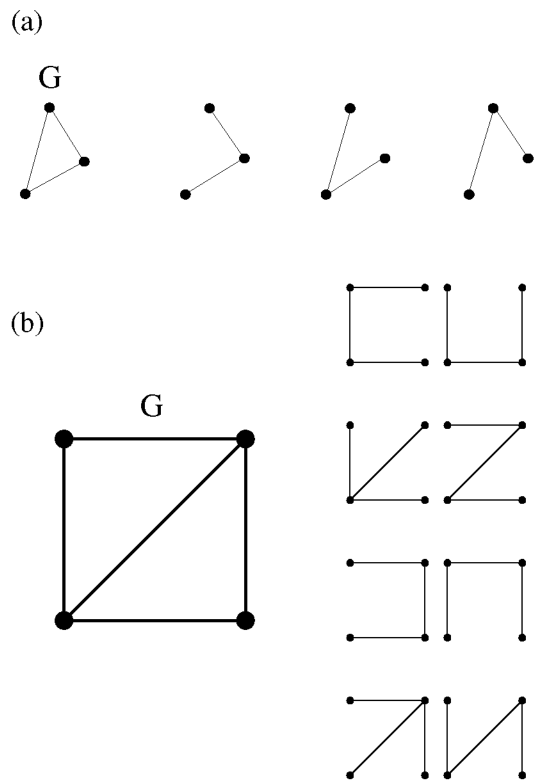


Figure 4 – A graph G and all its spanning tree T_G, whose total number is $\mathcal{N}(T_G)$ — a spanning tree is a loop-free subgraph of G, for which $V(T_G) = V(G)$. (a) A triangle-like G for which $\mathcal{N}(T_G) = 3$, (b) a one-diagonal-square-like graph G with $\mathcal{N}(T_G) = 8$.

We organize the chapter as the following. In section 2.3 we briefly review the necessary basic concepts, definitions and important known results. Our main novel expressions, summarized in Theorem 2.4.2 and Corollary 2.4.2.1 (this later for the LGF), are demonstrated in section 2.4. In section 2.5 we present concrete examples, the eleven Archimedean and the martini and (4,8^2) covering/medial lattices. In section 2.6 we prove Theorem 2.6.2, a new finding relating in the thermodynamic limit the free energy of the RWLS to the eSTGF. The conclusion is drawn in section 2.7.

Certain key results used along our work, known by specialists, but usually either just mentioned or whose proof are found only very sketched or developed only for particular situations in the literature, are made rigorous and general in the Appendices. This may be considered a third contribution of this study.

2.3 GENERAL DEFINITIONS AND THE BASIC THEORY

We start summarizing only the basic aspects of arbitrary graphs which will be necessary for the present work (for a comprehensive introduction to graph theory and applications see, e.g., ref. [4]).

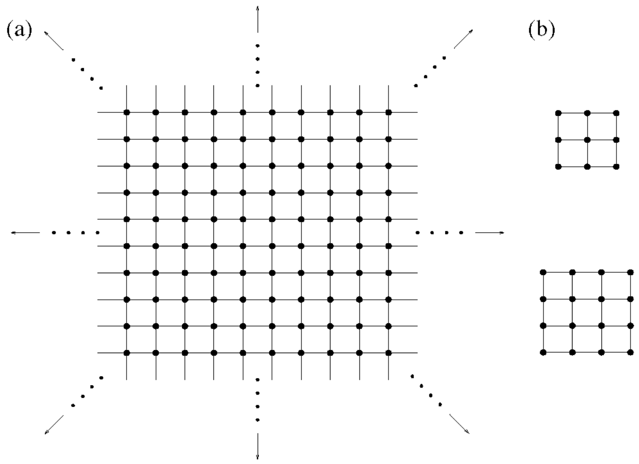


Figure 5 – (a) An illustration of a periodic infinite graph (or lattice) L: a 2D infinite square lattice. (b) Two examples, G_2 and G_3, of finite induced subgraphs of L — constituted by finite collections of vertices and edges of L — such that $\lim_{n\to\infty} G_n = L$.

A graph G is an ordered pair of disjoints sets (V, E). The order (size) of G is the cardinality $|V(G)|$ ($|E(G)|$) of its collection of vertices or sites $V(G)$ (edges or bonds $E(G)$). Let $V^{(2)}(G)$ be the set of unordered pairs of distinct elements of $V(G)$, i.e., $V^{(2)}(G) = \{(v_x, v_y) \equiv (v_y, v_x) \mid \forall\, v_x, v_y \in V(G),\ v_x \neq v_y\}$. G is said undirected, simple and connected if $E(G)$ is a subset of $V^{(2)}(G)$ and $\forall\, v_x, v_y \in V(G)$, always there exists at least one collection of elements $v_1, v_2, v_3, \ldots, v_{\mathcal{M}}$ of $V(G)$ (where $\mathcal{M} = 1, 2, 3, \ldots$), such that

$$\{(v_x, v_1), (v_1, v_2), \ldots, (v_{\mathcal{M}-1}, v_{\mathcal{M}}), (v_{\mathcal{M}}, v_y)\}$$

is a subset of $E(G)$. We call the sequence $\{(v_x, v_1), \ldots, (v_{\mathcal{M}}, v_y)\}$ a "path" between v_x and v_y. The degree $deg(x)$ (or coordination number) of a vertex $v_x \in V(G)$ is the total number of elements of $E(G)$ in the form (v_x, v_m) for all v_m's distinct elements of $V(G)$. A graph G is q–regular if $deg(x) = q\ \ \forall\, v_x \in V(G)$. We define the adjacency matrix of G, A_G, by ($v_x, v_y \in V(G)$)

$$\begin{aligned} A_G(v_x, v_y) &= 1, \qquad \text{if } (v_x, v_y) \in E(G), \\ A_G(v_x, v_y) &= 0, \qquad \text{otherwise.} \end{aligned} \tag{2.1}$$

Hereafter we will consider only undirected, simple and connected G's, moreover with $deg(x)$ finite $\forall\, v_x \in V(G)$, i.e., locally finite graphs.

A spanning tree T_G of G is a loop-free subgraph connecting all vertices of G, Fig. 4. For a finite graph G, we denote by $\mathcal{N}(T_G)$ the finite total number of spanning trees of G. Now, assume L an infinite periodic graph (more formally defined in section 2.3.2 and in this work indistinguishable also called a periodic lattice) so that: (a) G_n is a finite induced subgraph[4] of L; (b) $|V(G_n)|$ and $|E(G_n)|$ increases monotonically with n; and (c) $\lim_{n\to\infty} G_n = L$ (for an illustration, see Fig. 5). Then, the following limit exists[25]

(and is independent on the choice of the growing sequences of subgraphs)

$$\lim_{n\to\infty}\frac{\ln[\mathcal{N}(T_{G_n})]}{|V(G_n)|}=\lambda_L \tag{2.2}$$

λ_L is known as the spanning tree constant of L. We remark that Eq. (2.2) can be generalized in terms of measures on G (Theorem 5.1 in [23]), in which case λ_L is called tree entropy.

A relevant question is how to obtain λ_L for different L's. A powerful approach to calculate λ_L — relying on the general concept of a generating function [64] — has been proposed in [26]. Indeed, it has been shown [26] that a spanning tree generating function STGF, $T(z)$, can be constructed for an important class of periodic lattices (next Sections). One of its instrumental properties is that $T(1)=\lambda_L$ (or, as we are going to see, more rigorously $T(z\to 1^-)=\lambda_L$).

Among our main goals in the present contribution, one is to extend the classes of L's for which one can define and explicit derive a STGF.

2.3.1 A simple random walk on a graph G

A simple random walk on a graph G is a Markov chain C_G: (i) whose elements (states) are constructed from the set of vertices $V(G)$ of G, so that a typical element of C_G is a set (or chain) $c^{(\mathcal{M})}_{v_y\,v_x}=\{v_x,v_1,\ldots,v_{\mathcal{M}},v_y\}$ formed by the successive vertices of a path (see the definition above) between v_x and v_y; and for which (ii) a number $0\leq p(m,m+1)\leq 1$ is ascribed to each *successive pair* of vertices v_m and v_{m+1} in any chain $c^{(\mathcal{M})}_{v_y\,v_x}$ of C_G. Generally, $p(x,y)$ is called the transition probability from v_x to v_y and given by

$$\begin{aligned} p(x,y)&=\frac{1}{deg(x)}, &&\text{if } (v_x,v_y)\in E(G),\\ p(x,y)&=0, &&\text{otherwise.} \end{aligned} \tag{2.3}$$

For $\mathcal{M}=n-1$ (with $n=1,2,\ldots$) and specified $v_x,v_y\in V(G)$, let $\mathcal{C}_n(y,x)$ represent the total number of distinct chains of C_G in the form $c^{(\mathcal{M})}_{v_y\,v_x}=\{v_x,v_1,\ldots,v_{\mathcal{M}},v_y\}$. Since G is simple, $\mathcal{C}_1(x,x)=0\ \forall v_x\in V$. For a graph G, we define $p_n(x,x_0)$ as the probability to get to the vertex v_x, leaving from the vertex v_{x_0}, after n steps. We have that

$$p_n(x,x_0)=\frac{\mathcal{C}_n(x,x_0)}{\sum_{v_y\in V(G)}\mathcal{C}_n(y,x_0)}. \tag{2.4}$$

Note that for finite n, $\mathcal{C}_n(y,x_0)<\infty$ is non-zero only for a finite number of v_y's. Moreover, for the particular case of G a q–regular graph, $p_n(x,x_0)=\mathcal{C}_n(x,x_0)/q^n$. If $n=0$, we set $p_0(x,x_0)=\delta_{x\,x_0}$, with δ the Kronecker's delta.

Considering Eq. (2.4), we define the probability generating function of a random walk from v_{x_0} to v_x as ($z \in (-1, 1)$)

$$P(x, x_0, z) = \sum_{n=0}^{\infty} p_n(x, x_0)\, z^n. \tag{2.5}$$

Using the shorthand notation $p_n(x_0) = p_n(x_0, x_0)$ and $P(x_0, z) = P(x_0, x_0, z)$, it follows that

$$P(x_0, z) = \sum_{n=0}^{\infty} p_n(x_0)\, z^n = 1 + \sum_{n=1}^{\infty} p_n(x_0)\, z^n, \tag{2.6}$$

from which obviously $P(x_0, 0) = 1$.

2.3.2 Certain classes of infinite periodic graphs and the structure matrix

For our purposes, it is more appropriate to follow the characterization of infinite periodic graphs considered in [23, 65]. L is an infinite d–periodic — or just periodic for short — graph (lattice) if: (1) its vertices are labeled in $Z^d \times S$, where $S = \{1, 2, \ldots, \mathcal{S}\}$, with $|S| = \mathcal{S}$ finite (thus, for $v_x \in V(L)$ we write $x = (k, s)$, with $k \equiv (k_1, k_2, \ldots, k_d)$, all the k_n's being integers and $s \in S$); (2) the adjacency matrix A_L of L has the following property

$$A_L((k, s), (l, t)) = A_L((k - l, s), (0, t)), \quad \forall\, k, l\, \in Z^d, \quad \forall\, s, t\, \in S. \tag{2.7}$$

Note that a periodic lattice does not need to be q–regular [66] (cf., the beginning of section 2.3).

The above general definition can be put in more concrete terms by realizing L as an embedding structure in the R^d (a procedure obviously relevant in the explicit construction of physics lattice models [48, 47]). For completeness, this is described in the Appendix A.

An important concept is that of vertex-transitive graphs (see, e.g., ref. [3]). An isomorphism between two graphs G and H is a bijection $\theta : V(G) \rightarrow V(H)$ preserving adjacency, that is, the vertices v_x and v_y are adjacent in G if and only if $\theta(v_x)$ and $\theta(v_y)$ are adjacent in H. An automorphism of a graph is the isomorphism $\theta : V(G) \rightarrow V(G)$. If v_x and v_y are two vertices of G and if there is an automorphism mapping v_x to v_y, then v_x and v_y are said to be *similar* vertices. If v_x and v_y are similar vertices, for all n we have that $\mathcal{C}_n(x, x) = \mathcal{C}_n(y, y)$. For L a periodic lattice, for all $k \in Z^d$ and $s \in S$ the vertices $(0, s)$ and (k, s) are similar. This is so because the map $\theta_k : Z^d \times S \rightarrow Z^d \times S$, defined by ($\forall\, l \in Z^d$)

$$\theta_k(l, s) = (l + k, s), \tag{2.8}$$

is an automorphism of L with $\theta_k(0, s) = (k, s)$. This is a direct consequence of Eq. (2.7).

A graph for which all vertices are similar is called a *vertex-transitive graph*. A vertex-transitive graph is always q–regular but the converse is not necessarily true. A periodic L can be vertex-transitive only if it is also q–regular. However, a q–regular periodic L does not need to be vertex-transitive. Also, vertex-transitive graphs may not be periodic (a standard example being a Cayley graph).

Let $(Z^d \times S, E)$ be a periodic q–regular lattice L, then the transition probability between the vertices (k,s) and (l,t) of L reads

$$p((k,s),(l,t)) = p((l,t),(k,s)) = \frac{A_L((k,s),(l,t))}{q}. \tag{2.9}$$

Hence $p((k,s),(l,t)) = p((k-l,s),(0,t))$. The function $\Gamma : Z^d \to [0,1]^{\mathcal{S}\times\mathcal{S}}$ is defined such that for $k \in Z^d$, the element $\Gamma_{st}(k)$ of the matrix $\Gamma(k) \in [0,1]^{\mathcal{S}\times\mathcal{S}}$ is given by

$$\Gamma_{st}(k) = p((k,s),(0,t)). \tag{2.10}$$

Γ is known as the transfer matrix of a simple random walk on L. Observe that $\Gamma(k)$ is not zero only for a finite number of k's.

The following results will be useful for some derivations in the next Secs. For Γ^{T} representing the transpose of Γ, once

$$\begin{aligned}
{\Gamma^{\mathsf{T}}}_{st}(k) &= \Gamma_{ts}(k) = p((k,t),(0,s)) = p((0,s),(k,t)) = p((-k,s),(0,t)) \\
&= \Gamma_{st}(-k),
\end{aligned}$$

then

$$\Gamma^{\mathsf{T}}(k) = \Gamma(-k). \tag{2.11}$$

Furthermore, from the definition of $\Gamma(k)$, the matrix $\Phi = \sum_{k\in Z^d}\Gamma(k)$ is doubly stochastic. But so, it is well known that the spectrum radius $\rho(\Phi) \leq 1$. In other words, by denoting the eigenvalues of Φ as ϕ_n, $n = 1,2,\ldots,\mathcal{S}$, then necessarily we have that $\phi_1 = 1$ is the eigenvalue of largest modulus and $|\phi_n| \leq 1,\ n = 2,3,\ldots,\mathcal{S}$.

Now, $\Lambda : R^d \to C^{\mathcal{S}\times\mathcal{S}}$ defined by (with all w_n's in $w \equiv (w_1,w_2,\ldots,w_d)$ reals)

$$\Lambda(w) = \sum_{k\in Z^d} \Gamma(k)\,\exp[i\,k\cdot w], \tag{2.12}$$

is the *structure matrix* of L (observe that Λ is 2π-periodic in w). Given the relation in Eq. (2.11), $\Lambda(w)$ is Hermitian, namely, $\Lambda^\dagger(w) = \Lambda(w)$ (for $\Lambda^\dagger$ the Hermitian adjoint of Λ). Also, because $|\Lambda_{st}(w)| \leq \sum_{k\in Z^d}\Gamma_{st}(k) = \Phi_{st}$ ($\forall\, s,t \in S$, $\forall\, w \in R^d$), the Theorem 8.1.18 in [67] guarantees that $\rho(\Lambda(w)) \leq 1\ \ \forall\, w \in R^d$.

2.3.3 Important known results for the probability generating function and the spanning tree constant of certain classes of L's

Finally we review fundamental results for some types of infinite periodic graphs (lattices). Although they are discussed in the pertinent literature (e.g., ref. [53]), certain aspects of their proofs and implications are found fragmented, scattered in different sources. Therefore, next we outline known key facts, compiled as three main Theorems. We also present a relatively more compact proof (for instance, than that in ref.[53]) for one of them, namely, for Theorem 2.3.1. Hereafter $B = [-\pi, \pi]^d$ and 1 will represent the $\mathcal{S} \times \mathcal{S}$ identity matrix ($\mathcal{S} = |S|$).

The following Theorem (see [53]) relates the transition probabilities and the probability generating function of a simple random walk on a periodic q–regular lattice L to the structure matrix of L. Due to its importance, a complete proof is given in the Appendix A.

Theorem 2.3.1 . *Let L be an infinite d–periodic q–regular lattice of structure matrices $\Lambda(w) \in C^{\mathcal{S}\times\mathcal{S}}$ for $w \in R^d$. Thus, $\forall\ k \in Z^d$, $\forall\ s,t \in S$, $\forall\ z \in (-1,1)$, it holds true that*

$$
\begin{aligned}
p_n((k,s),(0,t)) \quad &= \frac{1}{(2\pi)^d}\int_B \exp[-i\,w\cdot k]\,[\Lambda^n]_{st}(w)\,dw, \\
P((k,s),(0,t),z) \quad &= \sum_{n=0}^{\infty} p_n((k,s),(0,t))\,z^n \\
&= \frac{1}{(2\pi)^d}\int_B \exp[-iw\cdot k]\big[\big(1 - z\,\Lambda(w)\big)^{-1}\big]_{st}\,dw.
\end{aligned}
$$

For $k = 0$ and any $s \in S$, we define $p_n(0,s) = p_n((0,s),(0,s))$ and $P((0,s),z) = P((0,s),(0,s),z)$. If L is also vertex-transitive, then independent on $s \in S$

$$
p_n(0) = p_n(0,s), \qquad P(0,z) = P((0,s),z). \tag{2.13}
$$

The probability generating function $P(0,z)$ is also called the *lattice Green function* LGF of L. It has been studied for distinct lattices in different dimensions (see, e.g., refs. [54, 68]).

Here it is worth commenting on a particularly straightforward, but relevant, situation. When the vertex set of a periodic lattice $L = (Z^d \times S, E)$ is such that $S = \{1\}$, it implies that L is also vertex-transitive and trivially isomorphic to $D = (Z^d, E_{Z^d})$, where $((k,1),(l,1)) \in E \Leftrightarrow (k,l) \in E_{Z^d}$. The isomorphism is given by the simple map $(k,1) \in Z^d \times S \Leftrightarrow k \in V = Z^d$. Thence, the adjacency matrices $A_L \Leftrightarrow A_D$, where

$$
A_D(k,l) = A_D((k-l),0) \quad \forall \quad k,l \in Z^d. \tag{2.14}
$$

In this way, every graph with vertex set $V = Z^d$ is d–periodic and vertex-transitive (hence also q–regular) if the above relation is satisfied by its adjacency matrix (a result guaranteed by Eq. (2.8) if s assumes just a single value).

Further, for the lattice $D = (Z^d, E_{Z^d})$, the function (in opposition to a matrix) $\lambda : R^d \to C$, defined by (for all w_n's in $w \equiv (w_1, w_2, \ldots, w_d)$ reals) $\lambda(w) = \sum_{k\in Z^d} \mu(k)\exp[i\,k\cdot$

$w]$, is called the structure function of D, where $\mu(k) = p(k,0)$ with $p(k,l) = A_D(k,l)/q$. In this case, the Theorem 2.3.1 assumes the following form.

Theorem 2.3.2 *Let $D = (Z^d, E_{Z^d})$ be as previously defined, also having the structure function $\lambda(w) \in C$ for $w \in R^d$. Then, $\forall\ k \in Z^d$, $\forall\ z \in (-1,1)$,*

$$\begin{aligned} p_n(k,0) &= \frac{1}{(2\pi)^d} \int_B \exp[-i\, w \cdot k]\, \lambda^n\, dw, \\ P(k,0,z) &= \sum_{n=0}^{\infty} p_n(k,0)\, z^n = \frac{1}{(2\pi)^d} \int_B \frac{\exp[-iw \cdot k]}{\big(1 - z\,\lambda(w)\big)}\, dw. \end{aligned}$$

For $k = 0$, it reads $p_n(0) = p_n(0,0)$ and $P(0,z) = P(0,0,z)$.

Another important result, making use of the transfer matrix $\Lambda(w)$ of L, has been demonstrated in [25]. It states that

Theorem 2.3.3 *For L an infinite d–periodic q–regular lattice, its spanning tree constant is given by*

$$\lambda_L = \ln[q] + \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\Big[\mathit{det}(1 - \Lambda(w))\Big]\, dw. \tag{2.15}$$

Note that since a vertex-transitive infinite periodic graph is a particular case of the L above, the relations in Eq. (2.15) extend to such class of lattices as well. In special, the eleven (vertex-transitive) Archimedean lattices L_{Archim} depicted in Fig. 6 constitute all the possible convex vertex-uniform tiling of the plane by means of regular polygons. For this full set, the expressions for $q\,(1 - \Lambda(w))$ have been derived and the above integrals have been solved in refs.[25, 24]. It has lead to the spanning tree constants for all the L_{Archim} (see Table 1 in section 2.5).

Also, for the particular case of the lattices D above, the Theorem 2 leads to a simpler expression for λ_L, which has been an important motivation for the advancements in [26].

Theorem 2.3.4 *Let $D = (Z^d, E_{Z^d})$ be a lattice as in Theorem 2.3.1 Then, the spanning tree constant for D reads*

$$\lambda_D = \ln[q] + \frac{1}{(2\pi)^d} \int_B \ln[1 - \lambda(w)]\, dw. \tag{2.16}$$

Lastly, we address the STGF $T(z)$ for a periodic vertex-transitive lattice developed in [26]. It can be summarized through the following theorem.

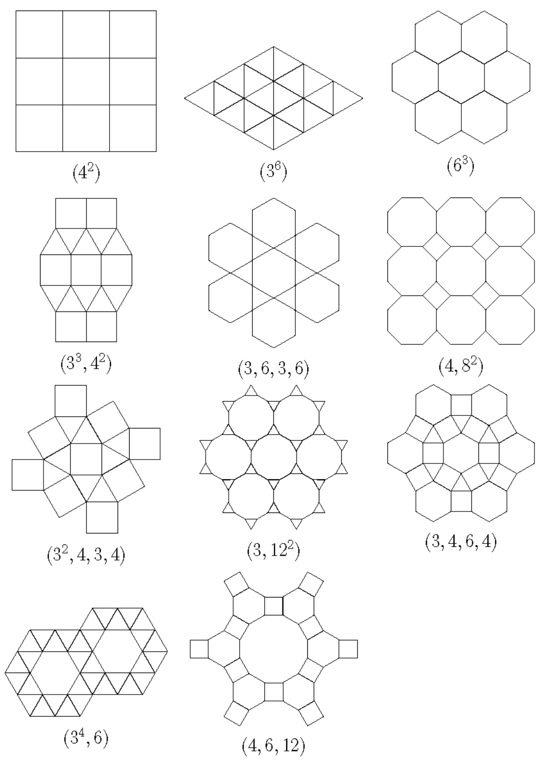


Figure 6 – The eleven bidimensional ($d = 2$) Archimedean infinite periodic vertex-transitive lattices. The labels $(a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots)$ follow the nomenclature in ref.[1] and indicate, in a cyclic order, all the polygons which meet at any arbitrary vertex of the graph.

Theorem 2.3.5 *Let L be an infinite d–periodic and vertex-transitive (hence q–regular) lattice. Assume $P(0,z)$ its LGF (or equivalently, the probability generating function of a simple random walk on L) and $p_n(0)$ as given before. For $0 < z < 1$, the spanning tree generating function STGF $T(z)$ of L is defined by*

$$T(z) = \ln[q] - \ln[z] - \int_0^z \frac{\big(P(0,u) - 1\big)}{u}\, du. \tag{2.17}$$

Then, the above $T(z)$ is the solution of the differential equation

$$-z\frac{dT(z)}{dz} = P(0,z) = \sum_{n=0}^{\infty} p_n(0)\, z^n, \tag{2.18}$$

with the boundary condition $T(z \to 1^-) = \lambda_L$.

Proof. The proof is straightforward using the fact that

$$\lambda_L = \ln[q] - \sum_{n=1}^{\infty} \frac{p_n(0)}{n} = \ln[q] - \int_0^1 \frac{\big(P(0,u) - 1\big)}{u} du,$$

obtained in [23].

We should observe that the more rigorous condition of $T(z \to 1^-) = \lambda_L$ above is justified since in principle $P(0,z)$ in Eqs. (2.17) and (2.18) is defined only for $|z| < 1$ (cf., Theorem 2.3.1). However, in practice when one explicit calculates $T(z)$ and then make the direct substitution $z = 1$, the final $T(1)$ readily gives λ_L. Therefore, one can write $\lambda_L = T(1)$ but bearing this technical point in mind.

It is worth noticing that the introduction of the STGF in [26] seems to be motivated by some particular known cases [59, 60, 61, 62, 23], as well as by the great similarity between the expressions for LGFs and λ_L for L's like hypercubes (in nD) and triangular (in 2D) periodic lattices. For instance, take $(1-\lambda) \to (z^{-1} - \lambda)$ in Eq. (2.16) of Theorem 2.3.4. In the resulting equation apply $-z\,\partial/\partial z$. Then, the final expression yields exactly $P(0,0,z)$ of Theorem 2.3.2.

By means of direct calculations [26], $T(z)$ has been obtained for three Archimedean — square, triangular and honeycomb — lattices (the first three in Fig. 6), leading to the corresponding λ_L's from $T(1)$. We note the square and triangular are the only Archimedean examples in the form $D = (Z^d, E_{Z^d})$ (however, the honeycomb presents an important similarity with them as discussed in section 2.5.1).

To be able to write a STGF for a vertex-transitive lattice L in terms of the associated LGF — moreover having the very relevant property of $T(1) = \lambda_L$ — is certainly a fundamental result, relating aspects of graph theory to simple random walks. Nonetheless, from a more practical point of view, namely, as a handily technique to obtain $T(z)$ and λ_L, Theorem 2.3.5 may not be so easy to apply. It has been explicit mentioned in [26] that: "In most cases, particularly for lattices of dimension greater than 2 [when often $\mathcal{S} > 1$], the integral [connecting T and P] cannot be performed". The difficulty to perform such integration is not really due to the dimension d *per se*, but rather to the size $\mathcal{S}$ of the unit cell. For example, in [26] the STGF for arbitrary $d-$dimensional simple hypercubes have been directly derived since $\mathcal{S} = 1$. However, the explicitly studied L's with $\mathcal{S} = 2$ (regardless of d) were only those whose particular symmetry features allowed close similarities to lattices of $\mathcal{S} = 1$ (e.g., see the hexagonal L in section 2.5.1). Our $T_e(z)$ next circumvents such problem once instead of computing more complicated integrals as $\mathcal{S}$ increases, we have to calculate determinants of larger $\mathcal{S} \times \mathcal{S}$ matrices.

But just to give a glance on the problem, we next explicit discuss $T(z)$ (using Eq. (2.17)) for the so called Archimedean cross lattice (4,6,12) — the last graph in Fig. 6 — whose unit cell has $\mathcal{S} = 12$ points and the coordination number is $q = 3$. The matrix $\Lambda(w_1, w_2)$ for this lattice is given in [24], or

$$\Lambda = \frac{1}{3}\begin{pmatrix} X & Y \\ Y^{\dagger} & X \end{pmatrix}, \tag{2.19}$$

with

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & e^{iw_1} & 0 & 0 \\ 0 & 0 & e^{iw_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{iw_2} \\ 0 & 0 & 0 & 0 & e^{iw_2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (2.20)$$

We calculate $P(0,z)$ from the Theorem 2.3.1. For so, we can choose any index $s \in \{1, 2, \ldots, 12\} = S$ since the (4,6,12) lattice is vertex-transitive. Choosing $s = 1$ (with $c = (2\pi)^{-2}\,(531441/59049)$)

$$\Big((1 - z\,\Lambda(w))^{-1}\Big)_{11} = (2\pi)^2\, c\, \frac{\mathcal{P}(z; w_1, w_2)}{\mathcal{Q}(z; w_1, w_2)},$$

for

$$\begin{aligned} \mathcal{P}(z; w_1, w_2) &= a_{10}\, z^{10} + a_8\, z^8 + a_6\, z^6 + 53946\, z^4 - 98415\, z^2 + 59049, \\ \mathcal{Q}(z; w_1, w_2) &= b_{12}\, z^{12} + b_{10}\, z^{10} + b_8\, z^8 + b_6\, z^6 + 728271\, z^4 - 1062882\, z^2 \\ &\quad +531441, \end{aligned}$$

where

$$\begin{aligned} a_{10} &= 8\,\sin[w_2 - w_1]\,\sin[w_2] - 10\,\cos[w_2 - w_1] - 10\,\cos[w_1] - 14\,\cos[w_2] \\ &\quad -17, \\ a_8 &= 216\,\cos[w_2 - w_1] + 36\,\cos[w_1] + 288\,\cos[w_2] + 801, \\ a_6 &= -486\,\cos[w_2 - w_1] - 162\,\cos[w_1] - 810\,\cos[w_2] - 11340, \end{aligned}$$

and

$$\begin{aligned} b_{12} &= (8\,(\cos[w_1]\,\cos[w_2] + \cos[w_1] + \cos[w_2]) + 12)\cos[w_1 - w_2] \\ &\quad +8\cos[w_1]\,\cos[w_2] + 12\,(\cos[w_1] + \cos[w_2]) + 13, \\ b_{10} &= -(144\,\cos[w_1 - w_2] + 468)\,(\cos[w_1] + \cos[w_2]) - 468\cos[w_1 - w_2] \\ &\quad -144\cos[w_1]\,\cos[w_2] - 918, \\ b_8 &= 4860\,(\cos[w_1 - w_2] + \cos[w_1] + \cos[w_2]) + 21627, \\ b_6 &= -8748\,(\cos[w_1 - w_2] + \cos[w_1] + \cos[w_2]) - 204120. \end{aligned}$$

Thus

$$P(0,z) = c\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathcal{P}(z; w_1, w_2)}{\mathcal{Q}(z; w_1, w_2)}\, dw_1 dw_2, \qquad (2.21)$$

and finally (inverting the integration order)

$$
\begin{aligned}
T(z) &= \ln[3] - \ln[z] - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathcal{I}(w_1, w_2)\, dw_1\, dw_2, \\
\mathcal{I}(w_1, w_2) &= c \int_0^z \frac{1}{u}\left(\frac{\mathcal{P}(u; w_1, w_2)}{\mathcal{Q}(u; w_1, w_2)} - \frac{1}{c}\right) du.
\end{aligned} \tag{2.22}
$$

In principle, the above integral in u could be solved by finding the roots of $\mathcal{P}$ and $\mathcal{Q}$, but which cannot be obtained through elementary methods because the polynomials degrees. Moreover, such roots should depend on involving trigonometric expressions of w_1 and w_2. Hence, the integrals in w_1 and w_2 seems to be too hard for a full analytic treatment (in fact, numerically it also would be a bit cumbersome, e.g., if the goal was to calculate only $T(1)$).

2.4 THE MAIN RESULTS

Here we present our main results. We introduce a (extended) spanning tree generating function eSTGF, $T_e(z)$, valid for infinite periodic q–regular lattices L, but which do not need to be vertex-transitive. Also, it obeys to $T_e(z \to 1^-) = \lambda_L$ (or for abuse of notation, $T_e(1) = \lambda_L$) and when particularized to vertex-transitive L's, $T_e(z)$ is somehow easily worked out than $T(z)$, thus providing simpler expressions for the STGF of graphs like all the Archimedean lattices.
The Lemma 2.4.1 below is necessary to prove fundamental properties of $T_e(z)$.

Lemma 2.4.1 *Let $L = (Z^d \times S, E)$ be an infinite d-periodic and q–regular lattice of structure matrix $\Lambda(w) \in C^{\mathcal{S}\times\mathcal{S}}$ for $w \in R^d$. Thus, $\forall\ z \in (-1, 1)$ it holds that*

$$
\frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \left[\sum_{s=1}^{\mathcal{S}} p_n(0, s)\right] \frac{z^n}{n} = -\frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\left[\mathit{det}(1 - z\,\Lambda(w))\right] dw. \tag{2.23}
$$

Above, $p_n(0, s)$ is the probability of a simple random walk on L to start at $(0, s)$ and to return to $(0, s)$ after n steps (Eq. (2.4)).

Proof. For any $s \in S$, by the Theorem 2.3.1 we have

$$
p_n(0, s) = p_n((0, s), (0, s)) = \frac{1}{(2\pi)^d} \int_B [\Lambda^n]_{ss}(w)\, dw.
$$

Since $|z| < 1$, the following is a convergent series (for Tr$[M]$ the trace of the matrix M)

$$
\frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \left[\sum_{s=1}^{\mathcal{S}} p_n(0, s)\right] \frac{z^n}{n} = \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \sum_{n=1}^{\infty} \int_B \frac{\mathrm{Tr}[(z\,\Lambda)^n(w)]}{n}\, dw. \tag{2.24}
$$

For $\Lambda(w)$ in section 2.3.2, $\rho(z\,\Lambda(w)) < 1$. Then, the Theorem A.2.1 in the Appendix A guarantees we can interchange the infinite sum with the integral in Eq. (2.24) and also that $\sum_{n=1}^{\infty} \text{Tr}[(z\Lambda)^n(w)]/n = -\ln[\text{det}(1 - z\,\Lambda(w))]$. Hence, Eq. (2.23) follows.

Now we prove the principal result of our work.

Theorem 2.4.2 *Let $L = (Z^d \times S, E)$ be as in Lemma 2.4.1. For L, we define the extended spanning tree generating function eSTGF, $T_e : (0,1) \to R$, as the well behaved series*

$$T_e(z) = \ln[q] - \ln[z] - \frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \left[\sum_{s=1}^{\mathcal{S}} p_n(0,s) \right] \frac{z^n}{n}. \tag{2.25}$$

Then:
(i) $T_e(z)$ can be cast as

$$= \ln[q] + \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\left[\frac{\textit{det}(1 - z\,\Lambda(w))}{z^{\mathcal{S}}} \right] dw; \tag{2.26}$$

(ii) $T_e(z)$ satisfies to

$$-z \frac{dT_e}{dz}(z) = \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} P((0,s), z), \tag{2.27}$$

with $T_e(z \to 1^-) = \lambda_L$; and
(iii) for L also vertex-transitive, $T_e(z) = T(z)$.

Proof. First, by Lemma 2.4.1 and Eq. (2.25) one has that $T_e(z)$ can also be written as

$$\begin{aligned} T_e(z) &= \ln[q] - \ln[z] + \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\left[\text{det}(1 - z\,\Lambda(w))\right] dw \\ &= \ln[q] + \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\left[\frac{\text{det}(1 - z\,\Lambda(w))}{z^{\mathcal{S}}} \right] dw, \end{aligned} \tag{2.28}$$

in agreement with Eq. (2.26).

Next, for $T_e(z)$ as in Eq. (2.25), take its derivative

$$-z \frac{dT_e}{dz}(z) = 1 + \frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \left[\sum_{s=1}^{\mathcal{S}} p_n(0,s) \right] z^n. \tag{2.29}$$

Now, since $p_0(0,s) = 1$, one gets $\mathcal{S}^{-1} \sum_{s=1}^{\mathcal{S}} p_0(0,s)\, z^0 = 1$ and from Eq. (2.29)

$$-z \frac{dT_e}{dz}(z) = \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} \left[\sum_{n=0}^{\infty} p_n(0,s) z^n \right] = \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} P((0,s), z). \tag{2.30}$$

Hence, Eq. (2.27) is verified. Furthermore, making $z \to 1^-$ in Eq. (2.28), one obtains exactly the first relation in Eq. (2.15) of Theorem 2.3.3, leading to λ_L. In this way, (ii) holds true.

Finally, if L is vertex-transitive, $P((0,s),z) = P(0,z)\ \forall s$. Therefore, Eq. (2.27) reduces to Eq. (2.18) and also considering that $T_e(1) = \lambda_L$, the statement (iii) follows.

The first identity in Eq. (2.28) is amazingly similar to the integral form of the spanning tree constant in Theorem 2.3.3. Such type of resemblance has already been observed in [26] for infinite periodic vertex-transitive graphs. At the same token, Theorem 2.4.2 (i) expresses $T_e(z)$ as an integral involving the matrix $\Lambda(w)$, analogously to how the Theorem 2.3.1 gives $P((0,s),z)$ in terms of an integral also involving $\Lambda(w)$. For L moreover vertex-transitive, it is worth contrasting Theorems 3 and 4. They describe two distinct approaches to get the same STGF $T(z)$. The workflow of Theorem 2.3.5 is $L \to \Lambda(w) \to P(0,z) \to T(z)$, whereas for Theorem 2.4.2 one has $L \to \Lambda(w) \to T(z)$. Hence, from Theorem 2.4.2 it is not necessary to calculate the function $P(0,z)$ to compute the STGF $T(z)$. Actually, in this case we can derive the eSTGF and then to use it to obtain the LGF through $P(0,z) = -z\, dT_e(z)/dz$.

Note that Theorem 2.4.2 implies that

$$\lambda_L = \ln[q] - \frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \left[\sum_{s=1}^{\mathcal{S}} p_n(0,s) \right] \frac{1}{n}. \tag{2.31}$$

This alternative series representation for λ_L can be obtained as an immediate consequence of Theorem 5.1 in [23]. As a last comment regarding eSTGF, we observe that for z small, we have from Eq. (2.25) $T_e(z) \approx \ln[q/z]$. This fact will be illustrated with numerical examples in Secs. 2.5.2 and 2.5.3.

Now we prove an important corollary of Theorem 2.4.2, leading to a new integral representation for the lattice Green function LGF of vertex-transitive lattices. It should be compared to Theorem 2.3.1.

Corollary 2.4.2.1 *Let $L = (Z^d \times S, E)$ be an infinite d–periodic, vertex-transitive (thus q–regular) lattice, then $\forall z \in (-1,1)$ the associated LGF $P(0,z)$ can be written as*

$$P(0,z) \;=\; 1 - \frac{z}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \frac{\partial[\mathit{det}(1 - z\,\Lambda(w))]/\partial z}{\mathit{det}(1 - z\,\Lambda(w))}\, dw. \tag{2.32}$$

Proof. For such L, $T_e(z) = T(z)\ \forall z \in (0,1)$. But in this case $P(0,z) = -z\, dT(z)/dz$. Using the first relation in Eq. (2.28), we directly get Eq. (2.32) for $z \in (0,1)$.

The integrand in Eq. (2.32) gives the finite result $-\text{Tr}[\Lambda(w)]$ when $z \to 0$. Thus, taking into account z multiplying the integral in Eq. (2.32), for $z = 0$ we find the correct $P(0,0) = 1$ (see Eq. (2.6)).

Finally, for $z \in (-1,0)$ let us define the well behaved real series

$$f(z) \equiv \ln[q] - \ln[-z] - \frac{1}{\mathcal{S}} \sum_{n=1}^{\infty} \Big[\sum_{s=1}^{\mathcal{S}} p_n(0,s) \Big] \frac{z^n}{n}. \tag{2.33}$$

By Lemma 2.4.1

$$f(z) = \ln[q] - \ln[-z] + \frac{1}{\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln[\text{det}(1 - z\,\Lambda(w))] dw. \tag{2.34}$$

So, for $z \in (-1,0)$ we have that $-z\, df(z)/dz$ is exactly the right hand side of Eq. (2.32). On the other hand, following very analogous steps to those used to prove Eq. (2.30) from Eq. (2.25), it reads

$$-z \frac{df}{dz}(z) = \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} P((0,s),z) = P(0,z). \tag{2.35}$$

Therefore, Eq. (2.32) is also true in the interval $(-1,0)$.

To conclude this section, it is instructive to consider the equivalence of the Corollary 2.4.2.1 with the Theorem 2.3.1 (for vertex-transitive lattices) by means of an example. For so, we choose the Archimedean kagome lattice, $(3,6,3,6)$ in Fig. 6. Its matrix $\Lambda(w_1,w_2)$ is given in [25], thus

$$1 - z\,\Lambda(w_1,w_2) = \begin{pmatrix} 1 & \frac{-z\,(1+e^{iw_2})}{4} & \frac{-z\,(1+e^{-iw_1})}{4} \\ \frac{-z\,(1+e^{-iw_2})}{4} & 1 & \frac{-z\,(1+e^{-i(w_1+w_2)})}{4} \\ \frac{-z\,(1+e^{iw_1})}{4} & \frac{-z\,(1+e^{i(w_1+w_2)})}{4} & 1 \end{pmatrix}.$$

Hence (for $\Delta = \cos[w_1] + \cos[w_2] + \cos[w_1 + w_2]$)

$$\mathcal{D}(z;w_1,w_2) \;\; \equiv \;\; \det(1 - z\,\Lambda(w)) - \frac{1}{16}\Big(z^3 + 6z^2 - 16 + (z^3 + 2z^2)\,\Delta\Big)$$

and (with $f_s(w_1,w_2) = 1 + \cos[\frac{1}{2}\,s\,(3-s)\,w_1 + (s-2)^2\,w_2]$, $s = 1,2,3$)

$$\Big((1 - z\,\Lambda(w))^{-1}\Big)_{ss} = \frac{1}{\mathcal{D}(z;w_1,w_2)} \left(1 - \frac{z^2}{8} f_s(w_1,w_2)\right).$$

From Theorem 2.3.1, $P(0,z)$ for the kagome lattice obeys to

$$P(0,z) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{(1 - z^2\, f_s(w_1,w_2)/8)}{\mathcal{D}(z;w_1,w_2)} dw_1 dw_2. \tag{2.36}$$

Equation (2.36) must lead to a same final expression for the LGF regardless the assumed value of $s = 1,2,3$ in the integration. And in fact, noticing the particular $\mathcal{D}$

dependence on w_1 and w_2 (in terms of Δ above), and once $f_3 = 1 + \cos[w_2]$, $f_2 = 1 + \cos[w_1]$ and $f_1 = 1 + \cos[w_1 + w_2]$, we find that $\forall\ s$ in Eq. (2.36), all the corresponding integrals are actually equivalent from the simple variables change: $(s = 3)$ $w_1 = v$, $w_2 = u$; $(s = 2)$ $w_1 = u$, $w_2 = v$; and $(s = 1)$ $w_1 = -v$, $w_2 = v + u$.

Now, taking the z derivative of $\det(1 - z\,\Lambda(w)) = \mathcal{D}(z; w_1, w_2)$, we obtain

$$z\,\frac{\partial \mathcal{D}(z; w_1, w_2)}{\partial z} = 3\,\mathcal{D}(z; w_1, w_2) - \sum_{s=1}^{3} \left(1 - \frac{z^2}{8}\, f_s(w_1, w_2)\right). \tag{2.37}$$

Considering Eq. (2.32) of Corollary 2.4.2.1 (with $\mathcal{S} = 3$), after some direct manipulations we get

$$P(0, z) = \frac{1}{(2\pi)^2}\,\frac{1}{3} \sum_{s=1}^{3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1 - \frac{z^2}{8}\, f_s(w_1, w_2)}{\mathcal{D}(z; w_1, w_2)} dw_1 dw_2. \tag{2.38}$$

Since, as previously shown, for each s the above integral yields the exact same result, Eq. (2.38) readily reduces to Eq. (2.36), as it should be.

2.5 $T_e(z)$ FOR DISTINCT LATTICES

Using the Theorem 2.4.2, we next provide the eSTGF for different examples. We first consider vertex-transitive L's (so, with the $eSTGF$ identical to the $STGF$ of Theorem 2.3.5), discussing the eleven Archimedean cases, Fig. 6. They represent all the possible combinations of regular polygons with every site being equivalent and uniformly tiling the plane (i.e., any vertex is surrounded by the same sequence of polygons, moreover with a translational symmetry). The λ_L's for these lattices have been calculated in [26] from $T(z = 1)$, but only for the square, triangular and honeycomb cases. For the other eight Archimedean lattices, the λ_L's have been directly obtained from Eq. (2.15) of Theorem 2.3.3 (see refs. [25, 24]).

In the sequel, we discuss two important non-vertex-transitive lattices (for which, obviously, there were no spanning tree generating functions previously calculated): the martini, whose spanning tree constant has already been derived in the literature; and the $(4, 8^2)$ covering/medial, for which λ_L, as far as we known, has not been reported anywhere.

2.5.1 The eSTGF (equivalent to the STGF) for the eleven Archimedean lattices

Table 1 – All the Archimedean lattices (ordered in increasing values of $\mathcal{S}$ and then of q, exactly as in Fig. 6). From the Bravais lattices, $\mathcal{S}$ is the number of points in an unit cell and q is the lattice coordination number. Here $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan constant and $\lambda_{tr} = \frac{3\sqrt{3}}{\pi}\left(1 - \sum_{n=0}^{\infty}\left(\frac{1}{(5+6n)^2} - \frac{1}{(7+6n)^2}\right)\right)$.

Square	(4^4)	1	4	$\lambda_{sq} = \frac{4G}{\pi}$ = 1.1662...
Triangular	(3^6)	1	6	λ_{tr} = 1.6153...
Honeycomb	(6^3)	2	3	$\frac{\lambda_{tr}}{2}$
Bridge	$(3^3, 4^2)$	2	5	1.4069258315...
Kagome	(3,6,3,6)	3	4	$\frac{\lambda_{tr}+\ln[6]}{2}$
Bathroom	(4,8^2)	4	3	0.7866842753...
Puzzle	(3^2,4,3,4)	4	5	1.4108556457...
Star	(3,12^2)	6	3	$\frac{\lambda_{tr}+\ln[15]}{6}$
Ruby	(3,4,6,4)	6	4	1.1448011236...
Maple Leaf	(3^4, 6)	6	5	1.3920235634...
Cross	(4,6,12)	12	3	0.7777955061...

To the best of our knowledge, the only Archimedean L's for which the STGF have been obtained are the square, triangular and honeycomb lattices, derived in [26] (and whose explicit integrations in w_1 and w_2 have lead to Generalized Hypergeometric Functions, GHFs). As already illustrated in the end of section 2.3.3, the difficulty for the others is that to obtain $T(z)$ from Eq. 2.18, one must first calculate $P(0,z)$ and then to perform the integral in Eq. (2.17), which can be a cumbersome task. On the other hand, for $T_e(z)$ in Eq. (2.25) no integration on the argument of P is required.

To calculate the eSTGF (here with eSTGF equals to the STGF) for the Archimedean lattices — see Table 1 — we have evaluated Λ using the definition in Eq. (2.12), with Γ from Eq. (2.10), and then employed the second equality of Eq. (2.25). For these lattices, we also have derived the LGF $P(0,z)$ considering the Corollary 2.4.2.1.

Next, we list just the final integral form expressions, resulting from above described intermediary computations. We should mention that for the square, triangular and honeycomb lattices, our $T_e(z) = T(z)$ (as well as the corresponding $P(0,z)$'s) completely agree with the same functions in [26]. In all which follows, $T_e(z) = T(z)$ and $P(0,z)$ are given in the compact forms

$$
\begin{aligned}
T(z) &= \ln\left[\frac{q}{z}\right] + \frac{1}{\mathcal{S}}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln[\mathcal{D}(z;w)]\,dw_1 dw_2, \\
P(0,z) &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathcal{E}(z,w)}{\mathcal{D}(z;w)}\,dw_1 dw_2,
\end{aligned}
\tag{2.39}
$$

where

$$
\mathcal{D}(z;w) = \det(1 - z\,\Lambda(w_1,w_2)), \quad \mathcal{E}(z;w) = -\frac{z^{\mathcal{S}+1}}{\mathcal{S}}\frac{\partial}{\partial z}\left(z^{-\mathcal{S}}\mathcal{D}(z;w)\right). \tag{2.40}
$$

For each Archimedean lattice, the values of q and $\mathcal{S}$ are listed in Table 1. For further compactness of notation, we set

$$
\Delta_0 = \cos[w_1] + \cos[w_2], \qquad \Delta_\pm = \cos[w_1 \pm w_2]. \tag{2.41}
$$

In this way, one finds

- Square-(4^4); $\mathcal{S} = 1, q = 4$:

$$
\begin{aligned}
\mathcal{D}(z;w) &= 1 - \frac{z}{2}\Delta_0, \\
\mathcal{E}(z;w) &= 1.
\end{aligned}
$$

- Triangular-(3^6); $\mathcal{S} = 1, q = 6$:

$$
\begin{aligned}
\mathcal{D}(z;w) &= 1 - \frac{z}{3}(\Delta_0 + \Delta_+), \\
\mathcal{E}(z;w) &= 1.
\end{aligned}
$$

- Honeycomb-(6^3); $\mathcal{S} = 2, q = 3$:

$$
\begin{aligned}
\mathcal{D}(z;w) &= -\frac{z^2}{3} + 1 - \frac{2z^2}{9}(\Delta_0 + \Delta_+), \\
\mathcal{E}(z;w) &= 1.
\end{aligned}
$$

- Bridge-$(3^3, 4^2)$; $\mathcal{S} = 2, q = 5$:

$$
\begin{aligned}
\mathcal{D}(z;w) &= -\frac{3z^2}{25} + 1 - \frac{4z}{5}\left(1 - \frac{z}{5}\Delta_+\right)\Delta_+ - \frac{2z^2}{25}(\Delta_0 + \Delta_+), \\
\mathcal{E}(z;w) &= 1 - \frac{2z}{5}\Delta_+.
\end{aligned}
$$

- Kagome-$(3,6,3,6)$; $\mathcal{S} = 3, q = 4$:

$$
\begin{aligned}
\mathcal{D}(z;w) &= -\frac{z^3}{16} - \frac{3z^2}{8} + 1 - \frac{(2+z)\,z^2}{16}(\Delta_0 + \Delta_+), \\
\mathcal{E}(z;w) &= -\frac{z^2}{8} + 1 - \frac{z^2}{24}(\Delta_0 + \Delta_+).
\end{aligned}
$$

- Bathroom-$(4,8^2)$; $\mathcal{S}=4, q=3$:

$$\begin{aligned}
\mathcal{D}(z;w) &= \frac{z^4}{81}-\frac{2z^2}{3}+1-\frac{4z^3}{27}\,\Delta_0-\frac{2z^4}{81}(\Delta_+ + \Delta_-),\\
\mathcal{E}(z;w) &= -\frac{z^2}{3}+1-\frac{z^3}{27}\,\Delta_0.
\end{aligned}$$

- Puzzle-$(3^2,4,3,4)$; $\mathcal{S}=4, q=5$:

$$\begin{aligned}
\mathcal{D}(z;w) &= a_0+a_1\,\Delta_0+a_2\,\Delta_0^2+a_3\,(\Delta_+ + \Delta_-),\\
\mathcal{E}(z;w) &= b_0+b_1\,\Delta_0+b_2\,(\Delta_+ + \Delta_-),
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{z^4}{625}-\frac{8z^3}{125}-\frac{2z^2}{5}+1,\\
a_1 &= -\frac{4z^4}{625}-\frac{8z^3}{125}-\frac{4z^2}{25},\\
a_2 &= \frac{4z^4}{625},\\
a_3 &= -\frac{12z^4}{625}-\frac{4z^3}{125},\\
b_0 &= -\frac{2z^3}{125}-\frac{z^2}{5}+1,\\
b_1 &= -\frac{2z^3}{125}-\frac{2z^2}{25},\\
b_2 &= -\frac{z^3}{125}.
\end{aligned}$$

- Star-$(3,12^2)$; $\mathcal{S}=6, q=3$:

$$\begin{aligned}
\mathcal{D}(z;w) &= \frac{4z^5}{81}+\frac{2z^4}{9}-\frac{4z^3}{27}-z^2+1-\frac{2(2z+3)\,z^4}{243}\,(\Delta_0+\Delta_+),\\
\mathcal{E}(z;w) &= \frac{2z^5}{243}+\frac{2z^4}{27}-\frac{2z^3}{27}-\frac{2z^2}{3}+1-2\left(\frac{z}{729}+\frac{1}{243}\right)z^4\,(\Delta_0+\Delta_+).
\end{aligned}$$

- Ruby-$(3,4,6,4)$; $\mathcal{S}=6, q=4$:

$$\begin{aligned}
\mathcal{D}(z;w) &= a_0+a_1(\Delta_0+\Delta_+)\\
&\quad +a_2(\Delta_+ + \Delta_- + 2\,\Delta_0\,\Delta_+ - (\Delta_+ + \Delta_-)\,\Delta_+),\\
\mathcal{E}(z;w) &= b_0+b_1(\Delta_0+\Delta_+),
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{z^6}{512}+\frac{3z^4}{32}-\frac{z^3}{16}-\frac{3z^2}{4}+1,\\
a_1 &= \frac{z^6}{512}-\frac{z^4}{32}-\frac{z^3}{16},\\
a_2 &= -\frac{z^6}{1024},\\
b_0 &= \frac{z^4}{32}-\frac{z^3}{32}-\frac{z^2}{2}+1,\\
b_1 &= -\frac{(3+z)\,z^3}{96}.
\end{aligned}$$

- Maple Leaf-$(3^4, 6)$; $\mathcal{S} = 6, q = 5$:

$$\begin{aligned}
\mathcal{D}(z;w) &= a_0 + a_1\,(\Delta_0 + \Delta_+) + a_2\,(\Delta_+ + \Delta_- + 2\,\Delta_0\,\Delta_+) \\
&\quad + a_3\,(\Delta_+ + \Delta_-)\Delta_+, \\
\mathcal{E}(z;w) &= b_0 + b_1\,(\Delta_0 + \Delta_+) + b_2\,(\Delta_+ + \Delta_- + 2\,\Delta_0\,\Delta_+),
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{13z^6}{15625} + \frac{24z^5}{3125} + \frac{27z^4}{625} - \frac{16z^3}{125} - \frac{3z^2}{5} + 1, \\
a_1 &= \frac{16z^6}{15625} - \frac{8z^5}{3125} - \frac{24z^4}{625} - \frac{8z^3}{125}, \\
a_2 &= -\frac{8z^6}{15625} - \frac{4z^5}{3125}, \\
a_3 &= \frac{4z^6}{15625}, \\
b_0 &= \frac{4z^5}{3125} + \frac{9z^4}{625} - \frac{8z^3}{125} - \frac{2z^2}{5} + 1, \\
b_1 &= -\frac{4z^5}{9375} - \frac{8z^4}{625} - \frac{4z^3}{125}, \\
b_2 &= -\frac{2z^5}{9375}.
\end{aligned}$$

- Cross-$(4, 6, 12)$; $\mathcal{S} = 12, q = 3$:

$$\begin{aligned}
\mathcal{D}(z;w) &= a_0 + a_1\,(\Delta_0 + \Delta_+) + a_2\,(\Delta_+ + \Delta_- + 2\,\Delta_0\,\Delta_+) \\
&\quad + a_3\,(\Delta_+ + \Delta_-)\,\Delta_+, \\
\mathcal{E}(z;w) &= b_0 + b_1\,(\Delta_0 + \Delta_+) + b_2\,(\Delta_+ + \Delta_- + 2\,\Delta_0\,\Delta_+),
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{13z^{12}}{531441} - \frac{34z^{10}}{19683} + \frac{89z^8}{2187} - \frac{280z^6}{729} + \frac{37z^4}{27} - 2z^2 + 1, \\
a_1 &= \frac{4z^{12}}{177147} - \frac{52z^{10}}{59049} + \frac{20z^8}{2187} - \frac{4z^6}{243}, \\
a_2 &= \frac{4z^{12}}{531441} - \frac{8z^{10}}{59049}, \\
a_3 &= \frac{4z^{12}}{531441}, \\
b_0 &= -\frac{17z^{10}}{59049} + \frac{89z^8}{6561} - \frac{140z^6}{729} + \frac{74z^4}{81} - \frac{5z^2}{3} + 1, \\
b_1 &= -\frac{26z^{10}}{177147} + \frac{20z^8}{6561} - \frac{2z^6}{243}, \\
b_2 &= -\frac{4z^{10}}{177147}.
\end{aligned}$$

For the above, we mention that by choosing different (but totally equivalent) unity cells for any lattice, the resulting Λ's at first sight may seem distinct. However, all can be transformed into each other through proper variable transformations upon integration, cf. Eq. (2.39). In particular, this is a procedure we have explicit used for the bridge and cross cases, seeking to symmetrise their final formulae. Thence, as it should be the $T(z)$ and $P(0,z)$ are unique.

From the functional form of the above $T(z)$'s, some interesting features can be identified. First, as previously remarked, the only Archimedean lattices classified as $D = (Z^d, E_{Z^d})$ (thus verifying Theorems 2.3.2 and 2.1) are the square and triangular. Nevertheless, although $\mathcal{S} = 2$ for the honeycomb, its $\mathcal{D}$ is similar to the triangular lattice. Moreover, the honeycomb is the only example in which $\mathcal{S} > 1$, but with $\mathcal{E} = 1$. So, in some aspects the honeycomb displays properties similar to a L having a structure function (section 2.3.3) rather than a structure matrix. This allows to construct $T(z)$ for the honeycomb through a direct analogy with the square and triangular lattices, what has been done in [26].

Second, the $\mathcal{D}$ for the bridge has the most uneven dependence on the Δ's (because Δ_+). It could indicate a certain spatial asymmetry for this lattice, which indeed is confirmed from the analysis of the Archimedean lattices unit cells presented in [24]. The bridge is the only example displaying an unbalanced (regarding the x and y directions) basic structure which once translated tiles the entire plane.

Third, as we have already mentioned, it is not our goal in the present contribution to work out in details the obtained T_e's (for instance, trying to write them in terms of special functions). But as proposed in [26], to view $T_e(z)$ in Eq. (2.25) as the logarithmic Mahler measure (see, e.g., refs. [69, 70, 71, 72]) of the proper Laurent polynomial related to det$(1 - z\,\Lambda)$ constitutes an important way to look for analytical expressions for the eSTGF. Actually, such idea has made possible [26] to express $T(z)$ for the square, triangular and honeycomb — as well as for other L's in higher dimensions — in terms of GHFs. Therefore, given that the $\mathcal{D}$ dependence on the Δ's, Eq. (2.41), for the honeycomb, kagome, and star, is exactly the same than that for the triangular lattice, the solution for this latter in [26] simply extends to the former three cases (observe also that the λ_L for these four lattices are closely associated, Table 1). Similarly, an eventual closed analytical expression for the ruby should be naturally extendable to the maple leaf and cross Archimedean L's.

2.5.2 The eSTGF for the martini lattice

The martini lattice in Fig. 7 — introduced in [1] — is an important network in the study of inhomogeneous site percolation [73]. Together with the honeycomb, it has been used [74] to accurately estimates the bond percolation of the kagome and star-$(3, 12^2)$ lattices. Also, the $O(n)$ loop model on the martini allows an exact derivation of critical points [75]. Its spanning tree constant has been obtained in [51].

The martini lattice is a 2–periodic regular graph. To realize it is not vertex-transitive, observe from Fig. 7 that $\mathcal{C}_3((0,1),(0,1)) = 0$, thus distinct from $\mathcal{C}_3((0,2),(0,2)) = 2$. Hence, for this graph one needs to consider the eSTGF instead of the STGF. In every unit cell there are $\mathcal{S} = 4$ points and the coordination number is $q = 3$. Its structure matrix is given by

$$\Lambda(w_1, w_2) = \frac{1}{3}\begin{pmatrix} 0 & 1 & e^{iw_2} & e^{iw_1} \\ 1 & 0 & 1 & 1 \\ e^{-iw_2} & 1 & 0 & 1 \\ e^{-iw_1} & 1 & 1 & 0 \end{pmatrix}.$$

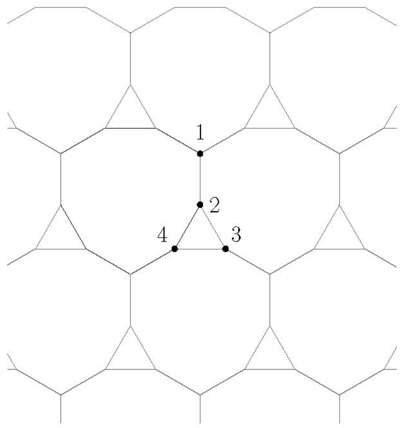


Figure 7 – The martini lattice, whose unit cell has 4 vertices (as indicated).

We have that

$$\begin{aligned} \det(1 - z\,\Lambda(w)) &= \frac{6\,(z+3)\,z^3}{81}\Big(g(z) - \frac{1}{3}(\Delta_0 + \Delta_-)\Big), \\ g(z) &= \frac{z^3 - 5z^2 - 3z + 9}{2z^3}. \end{aligned}$$

Thus, from Eq. (2.25)

$$T_e(z) = \frac{1}{4}\ln\left[\frac{6\,(z+3)}{z}\right] + \frac{1}{4}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln\left[g(z) - \frac{1}{3}\,(\Delta_0 + \Delta_+)\right] dw_1\,dw_2,$$

where the substitution of Δ_- by Δ_+ in the above integral has been achieved from a trivial change of variable. Note that Eq. (2.5.2) can be directly associated to the STGF of the Archimedean triangular lattice (previous section), with $g(z)$ here playing the role of $1/z$ in that expression. Therefore, following the exact same procedure of ref.[26], one can write the $T_e(z)$ for the martini lattice in terms of GHFs. For the present example we just observe that for z close to zero (recall we are assuming $z > 0$), $T_e(z) \approx \ln[3/z]$ and that the plot of $T_e(z)$ (numerically calculated) for $0 < z < 1$ is depicted in Fig. 8.

Finally, one can check that the spanning tree constant is

$$\begin{aligned} T_e(1) &= \frac{3}{4}\ln[2] + \frac{1}{4}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln[3 - \Delta_0 - \Delta_+]\,dw_1\,dw_2 \\ &= \frac{1}{4}(\ln[4] + \lambda_{tr}) = 0.7504060243\ldots, \end{aligned}$$

hence in full agreement with [51].

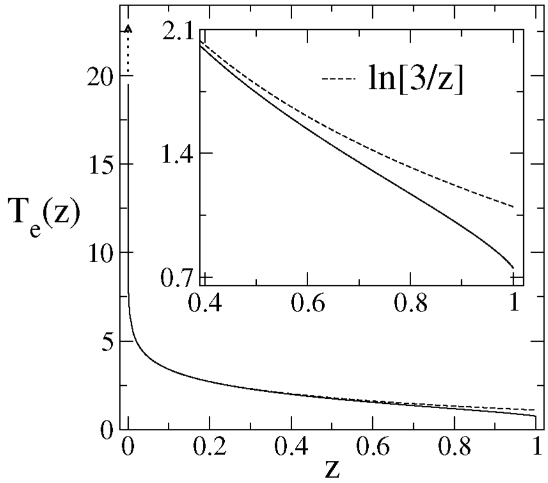


Figure 8 – The eSTGF for the martini lattice (calculated numerically). Note that $\ln[3/z]$ is a good approximation for $T_e(z)$ for z until around 0.5 (their difference is of about 31.7% for $z = 1$, see inset).

2.5.3 The eSTGF for the $(4, 8^2)$ covering/medial lattice

A challenging problem is that of determining the precise percolation thresholds in different graphs, for instance, in the Archimedean and related lattices [76, 77]. In this regard, a particularly interesting example is the $(4, 8^2)$ covering/medial, obtained from relatively simple edge transformations applied to the Archimedean bathroom-$(4, 8^2)$ (indeed, compare $(4, 8^2)$ in Fig. 6 with Fig. 9). It is also known as the square kagome [78], sharing several features with the Archimedean kagome-$(3, 6, 3, 6)$ lattice [79]. The bond percolation threshold and the critical point for the Potts model on the $(4, 8^2)$ covering/medial lattice have been studied in [80]. However, as far as we know, its spanning tree constant has not been analyzed in the literature.

The $(4, 8^2)$ covering/medial lattice is a 2-periodic regular graph which is non-vertex-transitive, Fig. 9. In every unit cell there are $\mathcal{S} = 6$ points and the coordination number of this lattice is $q = 4$.
From direct computations, we have that

$$\Gamma(0,0) = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}, \ \Gamma(1,0) = \begin{pmatrix} O & O \\ D & O \end{pmatrix}, \ \Gamma(0,1) = \begin{pmatrix} O & O \\ E & E \end{pmatrix}$$

$$\Gamma(-1,0) = \Gamma(1,0)^{\mathsf{T}}, \qquad \Gamma(0,-1) = \Gamma(0,1)^{\mathsf{T}}$$

where O is the 3×3 null matrix and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

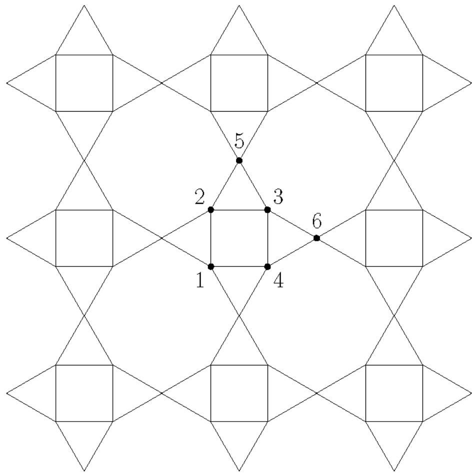


Figure 9 – The $(4, 8^2)$ covering/medial lattice. Its unit cell has 6 vertices (as shown).

So

$$\Lambda(w_1, w_2) = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 1 & e^{-iw_2} & e^{-iw_1} \\ 1 & 0 & 1 & 0 & 1 & e^{-iw_1} \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & e^{-iw_2} & 1 \\ e^{iw_2} & 1 & 1 & e^{iw_2} & 0 & 0 \\ e^{iw_1} & e^{iw_1} & 1 & 1 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} \det(1 - z\,\Lambda(w_1, w_2)) &= \frac{(z+2)^2\, z^4}{256}\left(g(z) + \frac{(z-4)}{z}\Delta_0 - \frac{1}{2}(\Delta_+ + \Delta_-)\right), \\ g(z) &= \frac{(8-z^2)(8-8z+z^2)}{z^4}. \end{aligned}$$

$$\begin{aligned} T_e(z) &= \frac{1}{3}\ln\left[\frac{4(z+2)}{z}\right] + \frac{1}{6}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln\Big[g(z) + \big(1 - \frac{4}{z}\big)\Delta_0 \\ &\quad -\frac{1}{2}(\Delta_+ + \Delta_-)\Big]\, dw_1\, dw_2. \end{aligned}$$

For z small, $T_e(z) \approx \ln[4/z]$. The plot of $T_e(z)$ (numerically calculated) for $0 < z < 1$ is shown in Fig. 10.

Lastly, the spanning tree constant of the $(4, 8^2)$ covering/medial lattice is given by

$$\begin{aligned} T_e(1) &= \frac{1}{3}\ln[12] + \frac{1}{6}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln\left[7 - 3\Delta_0 - \frac{1}{2}(\Delta_+ + \Delta_-)\right] dw_1\, dw_2 \\ &\approx 1.1217093(5)\ldots, \end{aligned}$$

where the last result has been obtained by numerical integration (using two distinct procedures, to double check for numerical accuracy).

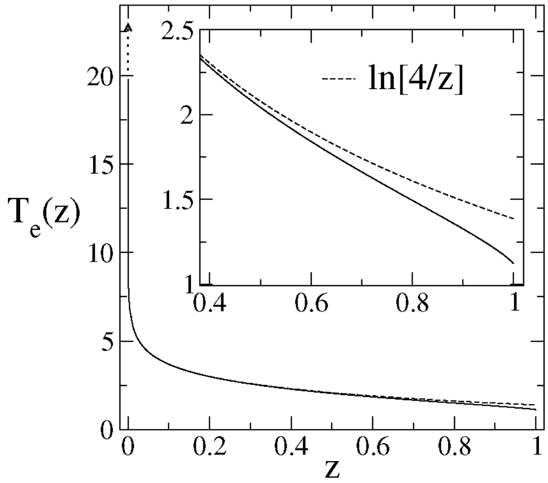


Figure 10 – The eSTGF for the $(4, 8^2)$ covering/medial lattice (calculated numerically). $\ln[4/z]$ is a good fit for $T_e(z)$ for z up to 0.6 (their difference is of about 19.1% for $z = 1$, see inset).

2.6 THE $eSTGF$ AND THE RANDOM WALK LOOP SOUP MODEL

As mentioned in the introductory section there is an important connection between spanning trees and lattice models in physics. In particular, as reviewed in [27], the spanning tree constant is known to bear a close relation with the critical temperature T_c in the Ising model. Moreover, it has been shown in [27] — using the STGF $T(z)$ — which for some simple vertex-transitive lattices (square and triangular), this type of relation is also valid at any temperature value.

So, a natural question is: there are also other physical models in which the STGF would map into some relevant quantity characterizing the system, furthermore this being the case for non-vertex-transitive graphs? As a final application for our present $T_e(z)$, we show the answer is positive for the random walk loop soup RWLS model.

The RWLS is a Poissonian ensemble of lattice loops. Generally, such type of model involves sets of one-dimensional loops living in higher dimensional spaces. The RWLS has been introduced in [38] as a proper discrete version of the Brownian loop soup. In fact, under Brownian scaling the former converges (in an appropriate sense) to the latter. This system has been extensively studied due to its relation with discrete Gaussian free fields [39] and with conformal loop ensembles [40]. Also, a non-backtracking loop soup has been developed in ref. [41], where it has been proved that the resulting model partition function is likewise associated with Gaussian free fields.

Below we present a brief review of the model. In the sequel, we show how an eSTGF can be used to construct the RWLS free-energy.

2.6.1 Basics of the RWLS model

Hereafter we closely follow the definitions in [44]. For $G = (V, E)$ an undirected, simple, connected and locally finite graph, a *closed walk* c of length $\ell_c = \mathcal{M} + 1$ ($\mathcal{M} = 1, 2, 3, \ldots$) is a closed chain in G (section 2.3.1), i.e., $c = c^{(\mathcal{M})}_{v_x\, v_x} = \{v_x, v_1, v_2, \ldots, v_{\mathcal{M}}, v_x\}$. The reversal of c is denoted as $c^{-1} = \{v_x, v_{\mathcal{M}}, v_{\mathcal{M}-1}, \ldots, v_1, v_x\}$. A cyclic shift operation σ_n (of order $0 \le n \le \mathcal{M}$) over the closed walk $c = \{v_x, v_1, v_2, \ldots, v_{\mathcal{M}}, v_x\}$ results in (where for $n = 0$, $v_0 = v_x$)

$$\sigma_n(c) = \{v_n, v_{n+1}, \ldots, v_{\mathcal{M}-1}, v_{\mathcal{M}}, v_x, v_1, v_2, \ldots, v_{n-1}, v_n\}.$$

For two closed walks c'' and c' such that $v_{x''} = v_{x'} = v_x$, we define their concatenation by

$$c'' \oplus c' = \{v_x, v_{1''}, v_{2''}, \ldots, v_{\mathcal{M}''}, v_x, v_{1'}, v_{2'}, \ldots, v_{\mathcal{M}'}, v_x\}.$$

The multiplicity m_c of a closed walk c is the largest number m such that c is the m-fold concatenation of some closed walk c' with itself, or $c = \bigoplus^m c'$.

For a closed walk c, the related *loop* l is the equivalent class of all the distinct cyclic shifts and associated inverses of c. Obviously, to c, c^{-1}, $\sigma_n(c)$, $[\sigma_n(c)]^{-1}$ $\forall n$ corresponds a same l. The length and multiplicity of a loop are defined as $\ell_l = \ell_c$ and $m_l = m_c$, for c any element of the equivalent class l. Note that given c we wave ℓ_c/m_c possible different cyclic shifts of this closed walk, each with its unique reverse. Thence, there are $|l| = 2\ell_l/m_l$ representative closed curves in the equivalent class l. We denote by $L(G)$ the set of all loops in G and by $L_r(G)$ the set of all loops in G with length r.

The weight of every loop l depends on a parameter z (for the range of z, see below) and is defined as

$$w(l, z) = \frac{z^{\ell_l}}{m_l}. \tag{2.42}$$

A loop configuration is a multiset [81] over $L(G)$, namely, a mapping

$a : L(G) \to N$, written as the formal linear combination

$$a = \sum_{l \in L(G)} \alpha_l\, l,$$

where $\forall\, l,\ \alpha_l \in N$. The set of all loop configurations $L_S(G)$ will be called a *loop soup*. Thus, we next can establish the weight of a, $w(a, z)$, as

$$w(a, z) = \prod_n \frac{w(l_n, z)^{\alpha_n}}{\alpha_n!}.$$

Now, for the remaining of this section we discuss the *partition function* $\mathcal{Z}_G(z)$ of the loop soup $L_S(G)$. For so, we are going to perform some formal manipulations, but which can be justified rigorously only for a finite G (although the final results might be valid for some infinite graphs [44]). Reference [44] properly mentions such fact, however with no further details. In the Appendix A we clarify this important technical point.

One can define the loop soup $L_S(G)$ partition function as

$$\begin{aligned} \mathcal{Z}_G(z) &\equiv \sum_{a\in L_S(G)} w(a,z) \\ &= 1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(l_1,\ldots,l_m)}\prod_{i=1}^{m} w(l_i,z). \end{aligned} \tag{2.43}$$

Let $L_r(m)$ be the set of all sequences $(l_1,\ldots,l_m)$ of loops satisfying

$$\ell_{l_1}+\ldots+\ell_{l_m}=r,$$

that is

$$L_r(m)=\bigcup_{r_1+\cdots+r_m=r} L(r_1,...,r_m), \tag{2.44}$$

where $L(r_1,...,r_m)=L_{r_1}(G)\times L_{r_2}(G)\times\ldots\times L_{r_m}(G)$. Then, by Eq. (2.43)

$$\mathcal{Z}_G(z)=1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{r=1}^{\infty}\sum_{(l_1,\ldots,l_m)\in L_r(m)}\prod_{i=1}^{m} w(l_i,z). \tag{2.45}$$

Equation (2.45) is defined for $z\in[0,1/k_{max}(G))$, where $k_{max}(G)$ is the maximum degree of G (see the Appendix A, Theorem A.3.1).

2.6.2 A novel formula for the free-energy of the RWLS model

Let $L=(Z^d\times S,E)$ be an infinite d–periodic q–regular lattice. We define $G_n=(V_n,E_n)$ as the vertex-induced subgraph of L (see section 2.3), determined by the vertex set

$$V_n=\{(k,s)\in Z^d\times S : k\in[-n,n]^d\}.$$

We can construct a well defined function $F:[0,1/q)\to R$, known as the *free-energy* of the RWLS in the thermodynamic limit, reading

$$F_L(z)\equiv-\lim_{n\to\infty}\frac{\ln\left[\mathcal{Z}_{G_n}(z)\right]}{|V_n|}. \tag{2.46}$$

The next relevant result for $F(z)$ is considered in the literature (see, e.g., [49] for the hypercube case), nonetheless whose general proof is not easily found in a single source. Because its importance for our purposes, we enunciate it here and give a complete proof in the Appendix A (for so we follow similar steps to those in [55], but which deals with non-backtracking walks in the context of the Ising model).

Theorem 2.6.1 *Let $L = (Z^d \times S, E)$ be an infinite d–periodic q–regular lattice and $\Lambda(w) \in C^{\mathcal{S}\times\mathcal{S}}$ its structure matrix. Then, for $z \in [0, 1/q)$ the free-energy $F_L(z)$ of the RWLS on L can be written as*

$$F_L(z) = \frac{1}{2\mathcal{S}} \frac{1}{(2\pi)^d} \int_B \ln\Big[\textit{det}(1 - z\,q\,\Lambda(w))\Big]\,dw. \tag{2.47}$$

Finally, we connect the eSTGF of the lattice L with the corresponding RWLS by means of the following new theorem.

Theorem 2.6.2 *Let $L = (Z^d \times S, E)$ be an infinite d–periodic q–regular lattice, whose $eSTGF$ is $T_e(z)$. For $F_L(z)$ the free-energy of the RWLS model on L, for all $z \in [0, 1)$*

$$F_L\Big(\frac{z}{q}\Big) = \frac{1}{2}\Big(T_e(z) - \ln\Big[\frac{q}{z}\Big]\Big)\,.$$

Proof. For $z \in (0, 1)$ the proof is a direct consequence of Eqs. (2.26) and (2.47). The case $z = 0$ can also be rigorously included if taking as the limit

$$F_L(0) = \lim_{z\to 0^+} F_L\Big(\frac{z}{q}\Big) = \frac{1}{2}\lim_{z\to 0^+}\Big(T_e(z) - \ln\Big[\frac{q}{z}\Big]\Big) = 0.$$

From Theorem 2.6.2 and Eq. (2.27) we have that $F_L(0) = 0$ and $\lim_{z\to 0} dF_L(z)/dz$ is finite, whereas

$$\lim_{z\to 1^-} F_L\Big(\frac{z}{q}\Big) = F_L\Big(\frac{1}{q}\Big) = \frac{1}{2}\Big(\lambda_L - \ln[q]\Big)$$

and

$$\lim_{z\to 1/q} \frac{dF_L}{dz}(z) = \frac{q}{2}\Big(1 - \frac{1}{\mathcal{S}}\sum_{s=1}^{s=\mathcal{S}} \lim_{z\to 1} P((0,s),z)\Big). \tag{2.48}$$

Therefore, it is natural to ask if the boundary point $1/q$ represents a critical point of the function $F_L(z)$, i.e., whether the limit in Eq. (2.48) does or does not exist.

For $d \geq 3$ ($d = 1$ or $d = 2$), a simple random walk on L is transient (recurrent) — see, e.g., refs. [6, 82]. In other words, for $d \geq 3$ ($d = 1$ or $d = 2$), $\lim_{z\to 1} P((0,s),z)$ is finite $\forall\, s \in S$ ($\lim_{z\to 1} P((0,s),z) = +\infty\ \forall\, s \in S$). So, when $d \geq 3$ ($d = 1$ or $d = 2$), the mentioned limit is finite (diverges). Hence, we conclude that a critical point or singularity appears only for the dimensions $d = 1$ and $d = 2$.

We illustrate this result with d-dimensional hypercubes (for $d = 1, 2, 3$), which are vertex-transitive lattices where $q = 2d$ and thus $\lim_{z\to 1/(2d)} dF_L(z)/dz = d(1 - \lim_{z\to 1} P(0,z))$. For such L's [26]

$$\Lambda_{\text{hypercube}}(w) = \frac{1}{d}\Big(\cos[w_1] + \cdots + \cos[w_d]\Big).$$

Then, from Theorem 2.3.1

$$P(0,z) = \begin{cases} \frac{1}{\sqrt{1-z^2}}, & d=1, \\ \frac{2}{\pi} K(z), & d=2, \\ \frac{4(1-9h^4)[K(4h^{\frac{3}{2}}/\sqrt{(1-h)^3(1+3h)})]^2}{\pi^2(1-h)^3(1+3h)}, & d=3, \end{cases}$$

where

$$h = \left(1+\sqrt{1-z^2}\right)^{-1/2} \left(1-\sqrt{1-\frac{z^2}{9}}\right)^{1/2}.$$

The $d=1$ expression comes straightforwardly from the integration of $(2\pi)^{-1}\int_{-\pi}^{\pi}(1 - z\cos[w_1])^{-1}\,dw_1$. The planar square lattice, $d=2$ case, is given in [53], with $K(\cdot)$ representing the complete elliptic integral of the first kind. Finally, for the cubic lattice, $d=3$, the above $P(0,z)$ has been derived in [83].

Observing that $\lim_{z\to 1} K(z) = \infty$, from the previous LGF, we have that $F_L(z)$ has a critical point at $z=1/2$ and $z=1/4$, respectively, for $d=1$ and $d=2$. On the other hand, since for $d=3$ [83, 84]

$$\begin{aligned} P(0,1) &= \frac{12}{\pi^2}\left(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right) \\ &\quad \times [K((2-\sqrt{3})(\sqrt{3}-\sqrt{2}))]^2 \\ &\approx 1.516386\ldots, \end{aligned}$$

we conclude that for the cubic lattice the boundary point $z=1/6$ is not a critical point.

Finally we should remark that for a non-backtracking loop soup (nbLS) model, the free energy for hypercube lattices has been obtained in [41]. In this case the critical points emerge also only when $d=1$ and $d=2$, at $z=1/(2d-1)$. However, the nbLS critical points at 1 ($d=1$) and $1/3$ ($d=2$) contrast with our respective values of 1/2 and 1/4 for the RWLS model.

2.7 CONCLUSION

In this contribution we have broadening the spanning tree generating function scope, first introduced in [26] for infinite periodic vertex-transitive lattices. Our $T_e(z)$ is: (i) valid for non-vertex-transitive regular L's, (ii) yields the spanning tree constant when evaluated at $z=1$, and (iii) reduces to the previous $T(z)$ in the vertex-transitive case.

The original $T(z)$ satisfies a differential equation involving the probability generating function (or lattice Green function LGF) of the corresponding lattice. In some contexts, however, the integration to get $T(z)$ can be a hard task (as we have seen through few examples). The eSTGF $T_e(z)$ can be obtained by integrating (over w)

$\ln[\det(1 - z\,\Lambda(w))/z^{\mathcal{S}}]$, with $\Lambda(w) \in C^{\mathcal{S}\times\mathcal{S}}$ the structure matrix of L. Such representation allows, for vertex-transitive lattices, to compute $T_e(z) = T(z)$ independently on the LGF $P(0,z)$. In this case our approach also allow to derive a new integral expression for $P(0,z)$.

As illustrations we have worked in details the eleven Archimedean (all vertex-transitive) and the non-vertex-transitive martini and $(4,8^2)$ covering/medial lattices. It is worth recalling that the purpose of this contribution was not to present an exhaustive list of examples. So, we have considered just planar ($d = 2$) L's. Surely, an interesting future development is to explore other lattices, in special those in three or more dimensions ($d \geq 3$) usually with higher $\mathcal{S}$ values. Regarding further generalizations of $T_e(z)$, currently we are studying the eventual relaxation of the q-regularity condition (results will be reported in the due course).

Finally, we have demonstrated that the eSTGF is, in the thermodynamic limit, essentially the free energy of the random walk loop soup model [38]. This provides a second example after the Ising model (for the square and triangular lattices [27]) — and a first for non-vertex-transitive L's — in which not only the spanning tree constant but also the full spanning tree generating function has a direct relation to key statistical physics quantities for lattice systems.

Of course, a very ambitious research project would be to determine which are the minimal necessary conditions for a physical discrete (lattice) model to be at least in part described by a eSTGF. We hope the present findings may motivate future investigations with such a goal.

3 THE EXACT SOLUTION OF THE ISOTROPIC ISING MODEL FOR THE ELEVEN ARCHIMEDEAN LATTICES IN TERMS OF THE CORRESPONDING SPANNING TREES GENERATING FUNCTIONS

3.1 ABSTRACT

We calculated the isotropic Ising free energy for the eleven Archimedean lattices and the martini lattice, and established some connections with the formulas for its $eSTGF$, that were obtained in chapter 2, the connections we found follows the work in [27] and they show that for these lattices, the Ising free energy can be obtained from the $eSTGF$ via a set of auxiliary functions, $\phi_1(K), .., \phi_{n_L}(K)$, where n_L is a positive integer that depends on the lattice. In the case $n_L = 1$ (square, triangle, hexagonal, kagome, Star, Martini) we obtained the additional property that

$$\phi(K_c) = 1,$$

where K_c is the critical point of the isotropic Ising free energy.

3.2 INTRODUCTION

The utmost approach towards describing matter in its condensed forms [85] involves the application of quantum many-body theory [86]. There are, however, phenomena whose proper understanding depends less on the detailed microscopic description of constituent interactions and more on the exact topological structure of matter organization, resulting in emergent properties and scaling behavior.

Indeed, the mathematical framework of the renormalization group [87] is spectacularly successful in predicting what happens near a continuous phase transition. Is conceptual pillars are scaling, renormalization and universality. Nonetheless, a too simplistic view on the problem may lead to wrong conclusions. For instance, somewhat naïve scaling assumptions led Landau to infer incorrect exponents for critical points [88]. The crucial mistake was to assume that a sequence of functions with a certain property — e.g., smoothness — has a limit that automatically inherits this property (that is why, in quantum mechanics for instance, valid states must belong to *bona fide* Hilbert spaces).

In this context, the *tour de force* Onsager's exact solution of the 2D ferromagnetic Ising model in 1942 (in zero field and for a square latticed), which he published in 1944 [28], was a breakthrough achievement (for a nice historical account see, e.g., [89]). His solution correctly predicted the scaling exponents and provided fascinating insights into what actually happens near the critical point. The ensuing advances also showed the value of closed analytic results. Since that time, a number of other solvable models have been studied, providing considerable insight into the macroscopic behavior of condensed matter systems [90, 91, 92, 93].

More broadly, exact expressions have provided especially useful information about the behavior of systems far from critical points, beyond the reach of renormalization group protocols (since scale invariance symmetry breaks down far from critical points). Frequently, only numerical solutions are used to investigate regimes far from critical points [94, 95].

In this context, exact solutions stand unparalleled in providing such insight.

The importance of topology in physics has been well known for centuries. Symplectic geometry and later differential topology are of fundamental importance to classical and relativistic mechanics, for instance. Not surprisingly, topology also plays a fundamental role in statistical mechanics. Topological data analysis, which has its roots in statistical mechanics, is a recent example.

In the 1960s, the statistical mechanics community was introduced to the methods of mathematical graph theory via the books of Harary, in particular, Graph Theory and Theoretical Physics. The chapter of Kastelyn, considered by many to be seminal, showed the connection between graph theory and exact enumeration problems in combinatorics, including a graph theoretical approach to the exact solution of the 2D Ising model. In fact, every planar lattice model can be treated this way.

Intuitively, a full periodic tiling (or tessellation) of the flat plane implies that it is possible to cover the entire Euclidean 2D space by means translations of a basic motif (or geometric shape) without leaving overlaps or gaps. The most symmetric tilings are those provided by regular convex polygons. The three so called regular tilings give rise to the simplest structures (for a formal definition see, e.g., [1]), resulting from the ordered repetition of either an equilateral triangle, a square, or a regular hexagonal. The second family in this hierarchical classification is formed by the eight semiregular tilings, whose fundamental shape is constituted by the juxtaposition of two or three regular convex polygons. They yield the bridge, kagome, bathroom, puzzle, star, ruby, maple leaf and cross lattices.

The above set of tessellations, depicted in Fig. 11, is known as the eleven Archimedean lattices. They exhaust all the possible combinations of regular polygons tiling the plane with the condition that every vertex is equivalent, namely, any vertex is surrounded by the same sequence of polygons, moreover with a translational symmetry [1]. The full equivalence of their vertices make Archimedean lattices very useful in treating a great diversity of distinct phenomena in physics [96, 97, 98, 99, 100, 101, 102, 103, 104, 105]. Actually, their great symmetry give rise to special features like isotropic structures in photonic crystals [106] and quantum states which allow universal quantum computation

[107], or yet being a key geometry in the architecture of protein containers in viruses [16].

In [27] was found a relation between spanning trees and the Ising model for the square lattice via a spanning tree generating function (STGF) defined in [26], this relations is valid for all temperatures T. Therefore, given the mentioned importance of Archimedean lattices, they seem a first natural class of infinite periodic graphs to study those relations, in trying to do so, we defined in chapter 2 the notion of an extended $STGF$ ($eSTGF$) that allows also regular non-vertex transitive lattices. If the lattice is vertex transitive then $STGF = eSTGF$. We calculated the $eSTGF$ for all the archimedan lattices and for the martini lattice (Fig. 7) in chapter 2. In this chapter we will show some relations between the $eSTGF$ and the Ising model of all Archimedean lattices and the martini lattice. We also calculated the Ising model for all these lattice since in the literature, the only solutions reported out of this set are only six, namely, for the square, triangle, hexagonal, kagome, star and bathroom lattice. So in this chapter we also report for the first time the Ising free energy for the other 5 archimedian lattice and for the martini lattice.

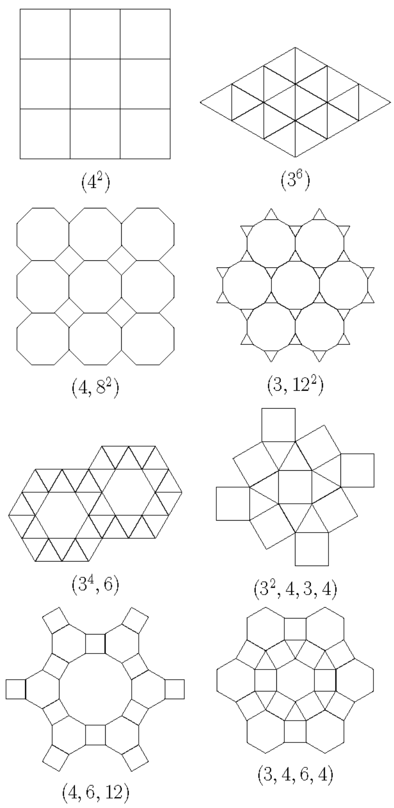


Figure 11 – The eleven planar Archimedean lattices. The nomenclature $(a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots)$ is that defined in [1] and indicates cyclically all the polygons which meet at any lattice arbitrary vertex.

3.3 BASIC CONCEPTS

3.3.1 Graphs

A graph G is an ordered pair $(V(G), E(G))$ consisting of two sets, of vertices $V \neq \emptyset$ and of edges E, together with an *incidence function* ϕ_G, which associates each edge of E to unordered pair of vertices of V (whenever the dependence on G is apparent, we just write V and E for short). The number of elements of V and E are denoted by $|V|$ and $|E|$ (throughout this work $|X|$ will represent the cardinality of the set X). If e is an edge and u and v are vertices such that $\phi_G(e) = \{u, v\}$, then e is said to join u and v. In this case, u and v are called adjacent vertices as well as “extremes” of e. An edge with distinct (identical) extremes is a link (loop). Two or more edges with the same pair of extremes are parallel edges. The degree (or coordination number) q of a vertex $v \in V$ is the number of elements of E in the form $\{v, u\}$ with $u \in V$. A graph is locally finite if for any $v \in V$ the corresponding degree is always finite. A q-regular G is a graph whose all vertices have the same degree q. Finally, G is connected if *there are not* two disjoint sets V_a and V_b such that $V = V_a \cup V_b$ and $\{v_a, v_b\} \in E$ for any $v_a \in V_a$ and any $v_b \in V_b$. An empty (also known as fully disconnected) graph has no adjacent vertices, i.e., the edge set is empty. Examples of elementary graphs are given in Fig. 12.

Given a graph G (possibly with loops and parallel edges) embedded in a surface S such that each edge is a continuous curve, we can define two orientations for each edge obtaining the set of all directed edges $\vec{E}(G)$ (or simply $\vec{E}$), with $|\vec{E}| = 2|E|$. If $d \in \vec{E}$, then $d : [0, 1] \to S$ is a continuous curve whose origin is $d(0)$ and end $d(1)$. The inverse of d, $\overline{d} : [0, 1] \to S$, is defined as $\overline{d}(t) = d(1 - t)$; obviously of origin $\overline{d}(0) = d(1)$ and end $\overline{d}(1) = d(0)$.

A *simple* graph $G(V, E)$ has no loops or parallel links and allows a somewhat more straightforward description since one may dispense the concept of incident function ϕ_G. Indeed, the elements of E can generally be written as $\{u, v\}$, with $u, v \in V$ adjacent vertices and $u \neq v$. In a diagram of a simple G, the edges (or links) labels might then be omitted. Further, for (v, u) an ordered pair with $v, u \in V$, one can define $\vec{E}$ as

$$\vec{E} = \{(v, w) : \{v, w\} \in E\}. \tag{3.1}$$

If $d = (v, w) \in \vec{E}$, $d(0) = v$, $d(1) = w$ and $\overline{d} = (w, v) \in \vec{E}$.

3.3.2 Periodic and quotient graphs

Obviously, in physical models periodic lattices are fundamental. So, here it is convenient to introduce periodic graphs and quotient graphs in a bit more rigorous manner, but whose effort pays off due to their usefulness for our later purposes. Actually, these concepts can be formulated in very abstract — hence general — terms based

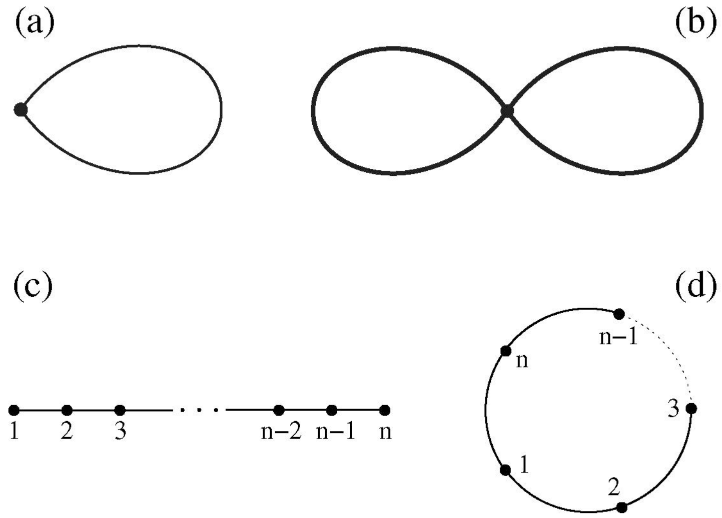


Figure 12 – Four very simple finite graphs: (a) a simple loop; (b) a double loop; (c) open chain of n vertices; (d) a cycle of n vertices.

on the idea of a d dimensional Z^d-action on a graph (see some basic properties in the Appendix B). However, next we shall avoid excessive technicalities, presenting a more direct definition particularized to the $d = 2$ case.

We first observe that generally an isomorphism between two graphs G and H is a bijection $\theta : V(G) \to V(H)$ preserving adjacency, that is, the vertices v and w are adjacent in G if and only if $\theta(v)$ and $\theta(w)$ are adjacent in H. An automorphism of a graph is the isomorphism $\theta : V(G) \to V(G)$.

A simple infinite graph G in R^2 is called 2-periodic if there are two independent vectors, v_1 and v_2 in R^2, such that

i) The translations by either v_1 or v_2 are automorphims of G.

ii) The number $\mathcal{S}$ of vertices of G in the parallelogram D spanned by the vectors v_1 and v_2, namely,
$$D = \{a\, v_1 + b\, v_2 : a, b \in [0, 1)\},$$
is finite. D is known as the fundamental region of G.

It is then a trivial fact that for any $(k, l) \in Z^2$, the map $t_{(k,l)}(v) = v + k\, v_1 + l\, v_2$ is an automorphism of G. We have that $(\{t_{(k,l)}\}, +)$, for "+" the usual sum of vectors, forms the translation group T_G of G. Examples of the fundamental domain D for three distinct infinite periodic graphs are depicted in Fig. 13. Note that the group T_G tiles the entire plane by acting on D.

We now recall that $n\, Z \oplus n\, Z$ (for a given $n \in N$) is the abelian additive group $\Lambda_n = (n\, Z \times n\, Z, +)$, where here the operation "+" is applied componentwise and the identity element is (0,0). Since Λ_n is a subgroup of Z^2 we can define a quotient graph

$G_n = G/\Lambda_n$ for each integer $n \geq 1$ (see Appendix B). $G_1 = G/\Lambda_1 = G/Z^2$ is called the fundamental quotient graph of G (illustrations in Fig. 14). The graph G_1 is finite and we have that our previously defined $\mathcal{S}$ is equal to $|V(G_1)|$ (Theorem B.1.2 in the Appendix B). If L is also a $q-$regular graph we have that (Theorem B.1.2)

$$\mathcal{E} = |E(G_1)| = \frac{q\,\mathcal{S}}{2}$$

.

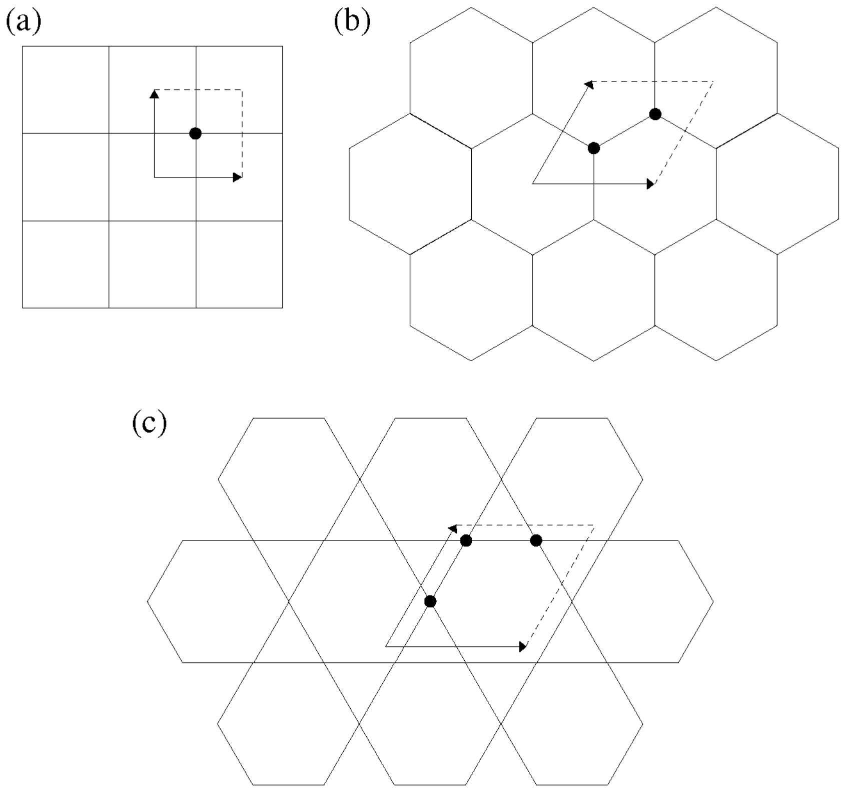


Figure 13 – Three examples of infinite periodic graphs and their fundamental domains D: (a) square, (b) hexagonal (honeycomb) and (c) kagome. The number of vertices, $\mathcal{S}$, within the fundamental domain is also highlighted.

Hereafter a planar (i.e., embedded in R^2), simple, connected, locally finite and 2-periodic graph, moreover with each face being a topological disc (thus non-generated, see [108] for details) will be called a lattice L.

Very relevant in present context is the so called *"Altered" Kac-Ward Matrix.* Let $G = (V, E)$ a finite graph embedded in an oriented torus T^2, such that the edges and faces of G are, respectively, rectilinear segments and topological discs. Assume γ_x and γ_y two simple closed curves in T^2, whose homology classes form a basis for the homology group of the torus. Consider also that this two curves avoid the vertices of G (in Fig. 15 we give an example with $G = L_{sq}/Z^2$, for L_{sq} denoting the square lattice). Thence, for each pair of complex numbers $z = (z_1, z_2) \in C^2$ the "Altered" Kac-Ward Matrix $\mathcal{K}(z; G)$ is a $2|E| \times 2|E|$ complex matrix indexed by the directed edges $\vec{E}(G)$,

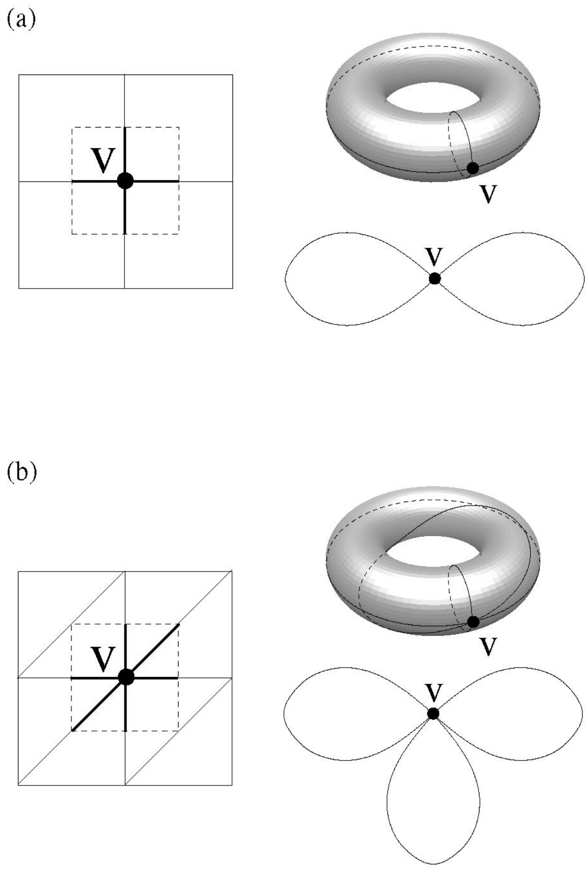


Figure 14 – Three different representations for the fundamental quotient graph of an infinite: (a) square G; (b) triangular G. In the grid representation, G/Z^2 corresponds to the dashed square (of equivalent parallel sides).

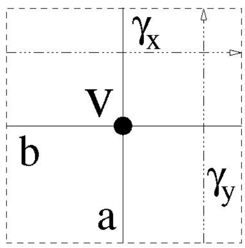


Figure 15 – An example of a finite graph, here $G = L_{sq}/Z^2$, embedded in the torus such that the simple closed curves γ_x and γ_y (the arrows indicate the positive orientations) do not cross any of the G vertices. As in Fig. 14, the parallel sides of the dashed square are equivalent (i.e., periodic boundary conditions).

reading [108]

$$\mathcal{K}(z_1, z_2; G)_{d_i,d_j} = \begin{cases} \alpha(z_1, z_2, d_i, d_j), & d_i(1) = d_j(0) \text{ and} \\ & d_j(1) \neq d_i(0) \\ 0, & \text{otherwise} \end{cases}, \tag{3.2}$$

where

$$\alpha(z_1, z_2, d_i, d_j) = (z_1)^{\gamma_x \cdot d_i} \, (z_2)^{\gamma_y \cdot d_j} \exp\left[\frac{i}{2}\,\theta(d_i, d_j)\right],$$

for $\theta(d_i, d_j) \in (-\pi, \pi)$ the oriented angle between the vectors d_i and d_j (see Fig. 16).

Furthermore, $\gamma_x \cdot d_i$ is 0 if d_i doesn't intersect γ_x, otherwise it is +1 (-1) if γ_x and d_i are both oriented either positively or negatively (have opposite orientations). The same applies to $\gamma_y \cdot d_j$. To illustrate this construction, from Fig. 15 we have that the angles for the Altered Kac-Ward matrix of the square lattice fundamental quotient graph are (here a_d, a_u, b_l and b_r means down, up, left and right regarding the vertex v in Fig. 15)

$$
\begin{aligned}
\gamma_x \cdot b_l &= \gamma_x \cdot b_r = \gamma_y \cdot a_d = \gamma_y \cdot a_u = 0, \\
\gamma_x \cdot a_d &= \gamma_y \cdot b_l = -1, \\
\gamma_x \cdot a_u &= \gamma_y \cdot b_r = +1, \\
\theta(b_r, b_r) &= \theta(b_l, b_l) = \theta(a_u, a_u) = \theta(a_d, a_d) = 0 \\
\theta(a_d, b_r), &= \theta(a_u, b_l) = \theta(b_r, a_u) = \theta(b_l, a_d) = +\frac{\pi}{2}, \\
\theta(a_d, b_l) &= \theta(a_u, b_r) = \theta(b_l, a_u) = \theta(b_r, a_d) = -\frac{\pi}{2}.
\end{aligned}
$$

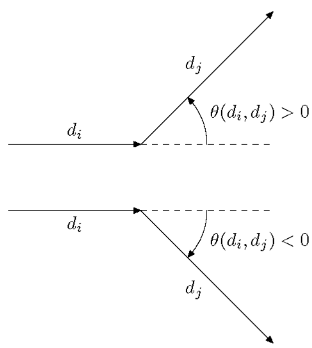


Figure 16 – For planar graphs whose edges are rectilinear segments, every directed d_i can be thought as a vector. So, if $d_i(1) = d_j(0)$ and $d_j(1) \neq d_i(0)$ there is a well defined oriented angle $\theta(d_i, d_j) \in (-\pi, \pi)$. Here two examples of such angles.

3.3.3 Spanning trees, even subgraphs and their generating functions

A spanning tree t_G of an arbitrary connected, simple and locally finite graph G is a loop-free subgraph connecting all vertices of G. Examples of which is and which is not a spanning tree of a graph G is illustrated in Fig. 17. For G furthermore finite, the total number of its spanning trees is denoted by $\mathcal{N}_t(G)$.

An important concept is that of spanning constant z_G for an infinite periodic graph G (see, e.g., [25]). In our present context, it is readily given by the well-behaved limit [25]

$$z_G = \lim_{n \to \infty} \frac{\mathcal{N}_t(G_n)}{V(G_n)}, \tag{3.3}$$

with G_n the n quotient graph of G.

For infinite periodic graphs G which are also vertex-transitive [3] (just the case of the Archimedean lattices) it has been proposed in [26] a Spanning Tree Generating Function (STGF) $\tilde{T}_G(z)$, written in terms of an integral over the corresponding lattice

Green's function [50, 54] and with the relevant property that $\tilde{T}_G(z = 1) = z_G$. The condition of vertex-transitivity has been relaxed to just q-regular G's in Theorem 2.4.2, yielding an expression $T_G(z)$ based on the structure matrix of G and hence easier to calculate. Furthermore, also $T_G(z = 1) = z_G$ and if G is vertex-transitive $T_G(z)$ reduces to $\tilde{T}_G(z)$. The explicit formula for $T_G(z)$ reads (Theorem 2.4.2)

$$T_L(z) = \ln[q/z] + \frac{1}{\mathcal{S}\, 4\,\pi^2} \int_B \ln\left[\mathcal{D}_L(z,w)\right]\, dw. \tag{3.4}$$

Details of Eq. (3.4) for the eleven Archimedean lattices will be presented in section 3.5.

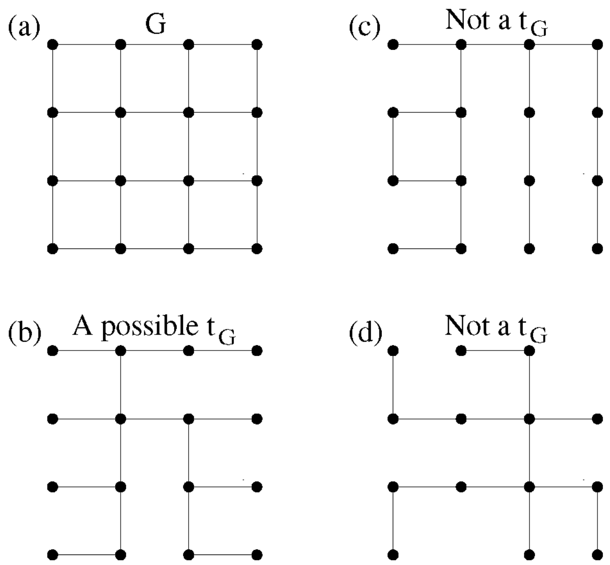


Figure 17 – (a) A graph G and (b) a possible spanning tree of it. A subgraph of G with (c) the edges forming loops or (d) not including all of its vertices is not a T_G.

We say that a subgraph H of $G = (E, V)$, with $V(H) = V(G)$ and $E(H) \subset E(G)$, is even if any $v \in V(H)$ has an even number of links (the number of loops can be arbitrary). The set of all even subgraphs of a graph G is denoted by $E(G)$. Note that an even subgraph does not need to be connected, see the example in Fig. 18.

Let $G = (E, V)$ be a finite graph, we define the Even Graph Generating Function (EGGF) of G as

$$\mathcal{F}(G,x) = \sum_{H \in E(G)} x^{|E(H)|} = \sum_{r=0}^{|E(G)|} \mathcal{N}(r)\, x^r, \tag{3.5}$$

where $\mathcal{N}(r)$ is the number of even subgraphs that have in total r edges (links plus loops). Since in any graph $G = (V, E)$ there is only one even graph with 0 edges, the empty graph, then $\mathcal{N}(0) = 1$.

It is worth illustrating Eq. (3.5) with the very simple examples of Fig. 12: (a) one vertex and one loop; (b) one vertex and two loops; (c) a finite open chain of n vertices; (d) a finite closed chain (cycle) of n vertices. For these graphs, all the even subgraphs and the corresponding $\mathcal{F}(G,x)$ are:

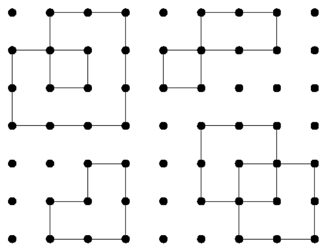


Figure 18 – Example of a disconnected even subgraph H of a finite 7×9 square graph G. Every vertex of H has even degree. All the vertices of G also belong to H.

(a) The trivial empty graph and G itself, hence

$$\mathcal{F}(G,x) = \sum_{r=0}^{1} \mathcal{N}(r)\, x^r = 1 + x.$$

(b) Since $V = \{v\}$ and $E(G) = \{e_1, e_2\}$, for both e_1 and e_2 loops, the even subgraphs are the empty graph, G, $(\{v\}, \{e_1\})$ and $(\{v\}, \{e_2\})$, with

$$\mathcal{F}(G,x) = \sum_{r=0}^{2} \mathcal{N}(r)\, x^r = 1 + 2x + x^2.$$

(c) Only the empty graph, so

$$\mathcal{F}(G,x) = \sum_{r=0}^{n-1} \mathcal{N}(r)\, x^r = 1.$$

(d) Similarly to (a), just the empty graph and G itself, thus

$$\mathcal{F}(G,x) = \sum_{r=0}^{n} \mathcal{N}(r)\, x^r = 1 + x^n.$$

An important limit associating EGGF and infinite lattices L has been proved in [108] (Lemma 4.3 in such Ref.). Let L be an infinite lattice as previously defined, also with all its edges being rectilinear segments. Hence, for $B = [0, 2\pi]^2$ and $G_n = L/\Lambda_n$, we have for each $x \in [0,1)$ that $(w = (w_1, w_2) \in B)$

$$\lim_{n\to\infty} \frac{\ln[\mathcal{F}(G_n, x)]}{n^2} = \frac{1}{8\pi^2} \int_B \ln\left[D_L(x,w)\right]\, dw, \qquad (3.6)$$

with (where $\mathcal{I}$ is the identity matrix)

$$\mathcal{D}_L(x,w) = \det[\mathcal{I} - x\,\mathcal{W}_L(w)] \qquad (3.7)$$

and (cf., Eq. (3.2))

$$\mathcal{W}_L(w) = \mathcal{K}(\exp[i w_1], \exp[i w_2]; L/Z^2). \qquad (3.8)$$

3.4 THE ISING MODEL.

The 2D Ising model in the thermodynamic limit and for a vanishing magnetic field $(B = 0)$ was solved by Onsager in 1944 [28] using quaternions. Kaufman reformulated the solution in terms of spinors in 1949 [109]. A different approach was introduced by Kac and Ward in 1952 [29], subsequently refined by Feynman [110] and Vdovichenko [111]. The Feynman-Vdovichenko method (FVM) relies on counting properly constructed graphs drawn on the same discrete lattice of the Ising model.

The partition function of the isotropic Ising model (with $B = 0$) on a finite (simple) graph $G = (E, V)$ is defined as $Z : [0, +\infty) \to R$, such that $(K = J(k_B T)^{-1})$

$$Z_G(K) = \sum_{\sigma \in \Omega} \prod_{\{u,v\} \in E} \exp[K\, \sigma_u\, \sigma_v], \tag{3.9}$$

where $\Omega = \{-1, 1\}^V$, i.e., Ω is the set of all functions (spin configurations) $\sigma : V \to \{-1, 1\}$.

The following result is well known and its proof can be found, e.g., in [112]. Let $G = (E, V)$ be a finite graph. Then, it holds for the Ising model partition function that

$$Z_G(K) = 2^{|V|} \cosh^{|E|}[K]\, \mathcal{F}(G, \tanh[K]). \tag{3.10}$$

The above function $\mathcal{F}$ is exactly the EGGF given in Eq. (3.5).

3.4.1 Thermodynamic limit and an exact general expression for the model free energy

Equation (3.9) has been defined for a finite graph G. Of course, the statistical physics interest relies on the model thermodynamic limit. To establish it, let L be a lattice as described above. So, the thermodynamic limit the free energy of the Ising model on L is given by

$$f_L(K) = \lim_{n \to \infty} \frac{\ln[Z_{G_n}(K)]}{|V(G_n)|},$$

where G_n is the quotient graph $G_n = L/\Lambda_n$.

The following expression provides a way to calculate the Ising model free energy for planar infinite periodic lattices L (see Theorem B.1.4 in the Appendix B)

$$\begin{aligned} f_L(K) = \ln[2] + \frac{\mathcal{E}}{\mathcal{S}} \ln[\cosh(K)] \\ + \frac{1}{\mathcal{S}\, 8\pi^2} \int_B \ln\left[D_L(\tanh[K], w)\right]\, dw. \end{aligned} \tag{3.11}$$

We shall remark that a formula equivalent to Eq. (3.11) has been reported in Refs. [111, 113]. It has been used to calculate the critical Ising temperature for all the eleven Archimedean lattices in Ref. [114]. However, despite the success in all these calculations, Eq. (3.11) has been proven to be rigorous (based on a method consisting of summing over random walk loops) only for the special case of the square lattice [55] . Therefore, as far as we know, the first general formal proof of Eq. (3.11) in the literature is that given in our Theorem B.1.4 in the Appendix B

Finally, from Theorem 1.1 and Proposition 2.2 in [108], it is straightforward to show that the Ising model critical point is given by $K_c =$ arctanh$[x_c]$, where x_c is the unique root in the interval (0,1) of the polynomial

$$\mathcal{P}_L(x) = \det[\mathcal{I} - x\,\mathcal{K}(0,0;L/Z^2)]. \tag{3.12}$$

3.5 THE ISING MODEL SOLUTIONS FOR ALL THE ELEVEN ARCHIMEDEAN LATTICES

Important parameters of the Archimedean lattices as well as $\mathcal{P}_L$ and the related values of x_c are listed in the Table 2. Further, by defining

$$\begin{aligned}
\Delta_\pm &= \cos[w_1 \pm w_2], \\
\Delta_0 &= \cos[w_1] + \cos[w_2], \\
\Delta_1 &= \cos[2w_1] + \cos[2w_2], \\
\Delta_2 &= \cos[2w_1 + w_2] + \cos[w_1 + 2w_2], \\
\Delta_3 &= \cos[2w_1 + 2w_2],
\end{aligned}$$

the expressions for $\mathcal{D}_L$ in Eq. (3.4) — for q and $\mathcal{S}$ see Table 2 — leading to the STGF for such lattices are summarized in Table 3 (for their derivations see Theorem 2.4.2).

To the best of our knowledge, six — out of eleven — exact solutions of the Ising model on Archimedean lattices (Fig. 11) have already been published in the literature [28, 32, 33]. They are the square, triangular, honeycomb, kagome, bathroom and star. Nonetheless, the other five, i.e., bridge, puzzle, ruby, maple leaf and cross, should still be calculated and we will do it shortly.
Now we fill this gap by deriving an exact integral representation of f_L for all the above L's, which is readily written down from the general expression in Eq. (3.11). For so, we determine L/Z^2 for each L and then explicitly construct $\mathcal{W}_L$, Eq. (3.8), to obtain Eq. (3.7). The necessary numerical constants $\mathcal{S}$, q and $\mathcal{E}$ are listed in Table 1 and in Table 2. The present full set of D_L's has not been reported anywhere. We also observe that as it should be, our expressions below for the previously known cases do agree with those in [28, 32, 33] after few algebraic manipulations [1].

Therefore, the Ising free energy f_L's for the eleven Archimedean lattices follow from Eq. (3.11) using the parameters in Table 2 and for (with $x = \tanh[K]$)

[1] Actually, from such careful comparison, we have spotted a simple sign misprint in f_L of the bathroom lattice in [33].

Table 2 – All the eleven Archimedean lattices in some of their features.

Lattice L	Label [1]	$\mathcal{E}$	n_L	The polynomial $\mathcal{P}_L(x)$ in Eq. (3.12) [a]	x_c [b]
Square	(4^4)	2	1	$(-1+2x+x^2)^2$	0.414214...
Triangular	(3^6)	3	1	$(1+x)^2(1-4x+x^2)^2$	0.267949...
Honeycomb	(6^3)	3	1	$(1-3x^2)^2$	0.577350...
Bridge	(3^3,4^2)	5	2	$(1-3x)^2(1+x)^2(1+x^2)^2$	1/3
Kagome	(3,6,3,6)	6	1	$(1+x)^4(1-2x-2x^3+x^4)^2$	0.435421...
Bathroom	(4,8^2)	6	2	$1-8x^3-2x^4+16x^6+8x^7+x^8$	0.601232...
Puzzle	(3^2,4,3,4)	10	3	$(1+x)^4(-1+2x+x^2+4x^3+9x^4-6x^5+7x^6)^2$	0.329024...
Star	(3,12^2)	9	1	$(1+x)^4(-1+2x-3x^2+2x^3+2x^4)^2$	0.670698...
Ruby	(3,4,6,4)	12	3	$(1+x)^4(1+x^2)^6(1-2x-2x^3+x^4)^2$	0.435421...
Maple Leaf	(3^4,6)	15	3	$(1+x)^8(1+3x^2)^2(-1+4x-7x^2+12x^3-3x^4+3x^6)^2$	0.344296...
Cross	(4,6,12)	18	3	$(1+2x^2+5x^4)^2(1-2x^2+2x^4-10x^6+x^8)^2$	0.616606...

[a] The $\mathcal{P}_L$ for the Kagome, Bathroom and Ruby lattices are different from those reported in [114]. However, as we have mentioned in the main text, the exact form of the polynomial depends on the embedding used for L. On the other hand, the value for x_c is always the same.

[b] x_c coincides with $\tanh[K_c]$ for K_c the critical point of the corresponding Ising model on L. Our x_c's values agree with [114].

Table 3 – All the eleven Archimedean lattices and the corresponding $\mathcal{D}_L$ in Eq. (3.4).

Lattice L	$\mathcal{D}_L$	Coefficients b_n
Square	$b_0+b_1\Delta_0$	$b_0=1,\ \ b_1=-\frac{z}{2}$
Triangular	$b_0+b_1(\Delta_0+\Delta_+)$	$b_0=1,\ \ b_1=-\frac{z}{3}$
Honeycomb	$b_0+b_1(\Delta_0+\Delta_+)$.	$b_0=-\frac{z^2}{3}+1,\ b_1=-\frac{2z^2}{9}$.
Bridge	$b_0+b_1\cos[w_1]+b_2(\cos[w_2]-\cos[2w_1]+\Delta_+)$	$b_0=1-\frac{z^2}{25},\ \ b_1=-\frac{2}{25}z^2-\frac{4}{5}z,\ \ b_2=-\frac{2}{25}z^2$.
Kagome	$b_0+b_1(\Delta_0+\Delta_+)$	$b_0=-\frac{z^3}{16}-\frac{3z^2}{8}+1,\ \ b_1=-\frac{(2+z)\,z^2}{16}$
Bathroom	$b_0+b_1\,\Delta_0+b_2(\Delta_++\Delta_-)$	$b_0=\frac{z^4}{81}-\frac{2z^2}{3}+1,\ \ b_1=-\frac{4z^3}{27},\ \ b_2=-\frac{2z^4}{81}$.
Puzzle	$b_0+b_1(\Delta_++\Delta_-)+b_2\Delta_0+b_3\Delta_1$	$b_0=\frac{z^4}{125}-\frac{8z^3}{125}-\frac{2z^2}{5}+1,\ \ b_1=-\frac{8z^4}{625}-\frac{4z^3}{125}$ $b_2=-\frac{4z^4}{625}-\frac{8z^3}{125}-\frac{4z^2}{25},b_3=\frac{2z^4}{625}$
Star	$b_0+b_1\,(\Delta_0+\Delta_+)$	$b_0=\frac{4z^5}{81}+\frac{2z^4}{9}-\frac{4z^3}{27}-z^2+1,b_1=-\frac{2(2z+3)\,z^4}{243}$.
Ruby	$b_0+b_1(\Delta_0+\Delta_+)+b_2(\Delta_-+\Delta_2)+b_3(\Delta_1+\Delta_3)$	$b_0=-\frac{3z^6}{2048}+\frac{3z^4}{32}-\frac{z^3}{16}-\frac{3z^2}{4}+1,$ $b_1=\frac{z^6}{1024}-\frac{z^4}{32}-\frac{z^3}{16}$ $b_2=-\frac{z^6}{1024},b_3=\frac{z^6}{2048}$.
Maple Leaf	$b_0+b_1(\Delta_0+\Delta_+)+b_2(\Delta_-+\Delta_2)+b_3(\Delta_1+\Delta_3)$	$b_0=-\frac{11z^6}{15625}+\frac{24z^5}{3125}+\frac{27z^4}{625}-\frac{16z^3}{125}-\frac{3z^2}{5}+1,$ $b_1=\frac{8z^6}{15625}-\frac{12z^5}{3125}-\frac{24z^4}{625}-\frac{8z^3}{125}$ $b_2=-\frac{8z^6}{15625}-\frac{4z^5}{3125},b_3=\frac{2z^6}{15625}$
Cross	$b_0+b_1(\Delta_0+\Delta_+)+b_2(\Delta_-+\Delta_2)+b_3(\Delta_1+\Delta_3)$	$b_0=\frac{5z^{12}}{177147}-\frac{34z^{10}}{19683}+\frac{89z^8}{2187}-\frac{280z^6}{729}+\frac{37z^4}{27}$ $-2z^2+1,$ $b_1=\frac{16z^{12}}{531441}-\frac{20z^{10}}{19683}+\frac{20z^8}{2187}-\frac{4z^6}{243},$ $b_2=\frac{4z^{12}}{531441}-\frac{8z^{10}}{59049},\ \ b_3=\frac{2z^{12}}{531441}$.

Table 4 – The coefficients a_n's for $\mathcal{D}_L$'s for all the Archimedean lattices.

<table>
<tr><th>Lattice L</th><th>Coefficients a_n</th></tr>
<tr><td>Square</td><td>$a_0 = (1+x^2)^2, \quad a_1 = 2\,x\,(-1+x^2).$</td></tr>
<tr><td>Triangular</td><td>$a_0 = (1+x)^2\,(1-2x+6x^2-2x^3+x^4),$
$a_1 = -2\,x\,(-1+x^2)^2.$</td></tr>
<tr><td>Honeycomb</td><td>$a_0 = 1+3x^4, \quad a_1 = 2\,x^2\,(-1+x^2).$</td></tr>
<tr><td>Bridge</td><td>$a_0 = (1+x)^2\,(1+x^2)^2\,(1-2x+5x^2),$
$a_1 = 2\,x\,(-1+x)\,(1+x)^2\,(2-x+5x^2+x^3$
$+x^4),$
$a_2 = 2\,x^2\,(-1+x^2)^3.$</td></tr>
<tr><td>Kagome</td><td>$a_0 = (1+x)^4\,(1-4x+10x^2-16x^3+22x^4$
$-16x^5+10x^6-4x^7+x^8),$
$a_1 = -2\,x^2\,(-1+x)^2\,(1+x)^4\,(1+x^2).$</td></tr>
<tr><td>Bathroom</td><td>$a_0 = (1+x^2)^2\,(1-2x^2+5x^4),$
$a_1 = 4x^3\,(-1+x^4),$
$a_2 = -2x^4\,(1-x^2)^2$</td></tr>
<tr><td>Puzzle</td><td>$a_0 = (1+x)^4\,(1+x^2)^2\,(1-4x+8x^2-4x^3$
$+2x^4+20x^5+32x^6-12x^7+21x^8),$
$a_1 = -4x^3\,(-1+x)^4\,(1+x)^6\,(1+x^2),$
$a_2 = 4x^2\,(-1+x)^3\,(1+x)^5\,(1+x^2)^2\,(1+3x^2),$
$a_3 = 2x^4\,(-1+x^2)^6.$</td></tr>
<tr><td>Star</td><td>$a_0 = (1+x)^4\,(1-4x+10x^2-16x^3+19x^4$
$-16x^5+10x^6-4x^7+4x^8),$
$a_1 = 2x^4\,(1+x)^4\,(-1+2x-3x^2+2x^3).$</td></tr>
<tr><td>Ruby</td><td>$a_0 = x^{24}+4x^{21}+6x^{20}+12x^{19}+26x^{18}+72x^{17}$
$+195x^{16}+312x^{15}+390x^{14}+624x^{13}+812x^{12}$
$+624x^{11}+390x^{10}+312x^9+195x^8+72x^7$
$+26x^6+12x^5+6x^4+4x^3+1,$
$a_1 = -4(x-1)^2x^3(x+1)^4(x^{12}+4x^{10}-x^9$
$+9x^8+x^7+4x^6+x^5+9x^4-x^3+4x^2+1),$
$a_2 = -4x^6\left(x^2-1\right)^6,$
$a_3 = 2x^6\left(x^2-1\right)^6$</td></tr>
<tr><td>Maple Leaf</td><td>$a_0 = 63x^{24}+456x^{23}+1578x^{22}+3768x^{21}$
$+7530x^{20}+13392x^{19}+20904x^{18}+28608x^{17}$
$+34899x^{16}+37968x^{15}+36204x^{14}+29808x^{13}$
$+21172x^{12}+13152x^{11}+7176x^{10}+3328x^9$
$+1341x^8+552x^7+186x^6+24x^5+18x^4$
$+16x^3+1,$
$a_1 = 4x^3\,(-1+x)^3\,(1+x)^8\,(2-4x+15x^2$
$-11x^3+27x^4-x^5+15x^6+11x^7+3x^8+5x^9$
$+2x^{10}),$
$a_2 = 4x^5\,(-1+x)^7\,(1+x)^9\,(1+x^2),$
$a_3 = -2x^6\,(-1+x^2)^9.$</td></tr>
<tr><td>Cross</td><td>$a_0 = 127x^{24}+384x^{22}+1188x^{20}+992x^{18}$
$+837x^{16}+324x^{14}+182x^{12}+24x^{10}+27x^8$
$+4x^6+6x^4+1,$
$a_1 = -4x^6\,(-1+x^2)^2\,(3+9x^2+31x^4+56x^6$
$+101x^8+93x^{10}+81x^{12}+10x^{14}),$
$a_2 = 4x^{10}\,(-1+x^2)^4\,(-2-3x^2-4x^4+x^6),$
$a_3 = 2x^{12}\,(-1+x^2)^6.$</td></tr>
</table>

3.6 RELATION BETWEEN THE FREE ENERGY OF THE ISING MODEL ON L AND THE STGF OF L

Here we prove our key finding, namely, that given a lattice L with a $STGF\ T(z)$, the free energy $f(K)$ of the Ising model on it can be exactly mapped to $T(z)$ if L is an Archimedean lattice.

For so, we start considering the following Laurent polynomials (for $z_1, z_2 \in C$ and n and m integers)

$$\hat{\delta}_{n\,m}(z_1, z_2) = \frac{1}{2}\Big(\frac{1}{z_1^n\, z_2^m} + z_1^n\, z_2^m\Big) \tag{3.13}$$

Observe that $\hat{\delta}_{1\,-1} = \hat{\delta}_{-1\,1}$ and for $\lambda_L = \exp[iw_l]$ we have

$$\hat{\delta}_{n\,m}(\exp[iw_1], \exp[iw_2]) = \cos[nw_1 + mw_2]. \tag{3.14}$$

and let $L = \{\hat{\delta}_{1\,0} + \hat{\delta}_{0\,1}, \hat{\delta}_{1\,0} + \hat{\delta}_{0\,1} + \hat{\delta}_{1\,1}, \hat{\delta}_{1 0}, \hat{\delta}_{0\,1} - \hat{\delta}_{2\,0} + \hat{\delta}_{1\,1}, \hat{\delta}_{1\,1} + \hat{\delta}_{1\,-1}, \hat{\delta}_{2\,0} + \hat{\delta}_{0\,2}, \hat{\delta}_{1\,-1} + \hat{\delta}_{2\,1} + \hat{\delta}_{1\,2}, \hat{\delta}_{2\,2} + \hat{\delta}_{2\,1} + \hat{\delta}_{1\,2}\}$. and for any $\hat{\delta}(z_1, z_2) \in L$, we define

$$\delta(w_1, w_2) = \hat{\delta}(\exp[iw_1], \exp[iw_2])$$

For convenience we also define $\hat{f}_L : [0, 1) \to R$ with

$$\begin{aligned} \hat{f}_L(x) \quad &= \quad \ln[2] - \frac{\mathcal{E}}{2\,\mathcal{S}} \ln[1 - x^2] \\ &\quad + \frac{1}{\mathcal{S}\,8\,\pi^2} \int_B \ln\left[D_L(x, w)\right]\, dw. \end{aligned} \tag{3.15}$$

So, trivially from $\cosh^2[K] = (1 - \tanh^2[K])^{-1}$ one gets $f_L(K) = \hat{f}_L(\tanh[K])$.

Now we present two formulas that will be the basis for our calculations. By considering the polynomials set L and the explicit expressions for the f's in section 3.5 and the T's in section 2.5.1, we have that for any Archimedean lattice L, there exists an integer $1 \le n_L \le 3$, two sequences of polynomials

$$\begin{aligned} a_L =& (a_0(x), ..., a_{n_L}(x)), \\ b_L =& (b_0(z), ..., b_{n_L}(z)), \end{aligned}$$

with $a_0(x), b_0(z) > 0$ for all $x, z \in [0, 1)$, and a sequence of Laurent polynomials in L,

$$\hat{\delta}_L = \{\hat{\delta}_0, ..., \hat{\delta}_{n_L}\}$$

such that $f_L(K) = \hat{f}_L(x)$ with $x = \tanh[K]$ and $T_L(z)$ are given by

$$\begin{aligned}\hat{f}_L(x) &= \ln[2] - \frac{\mathcal{E}}{2\mathcal{S}}\ln[1-x^2]\\ &+ \frac{1}{\mathcal{S}\,8\,\pi^2}\int_B \ln\left[a_0(x) + \sum_{i=1}^{n_L} a_i(x)\delta_i(w)\right]dw, \end{aligned}\tag{3.16}$$

$$\begin{aligned}T_L(z) &= \ln\left[\frac{q}{z}\right]\\ &+ \frac{1}{\mathcal{S}\,8\,\pi^2}\int_B \ln\left[b_0(z) + \sum_{i=1}^{n_L} b_i(z)\delta_i(w)\right]dw, \end{aligned}\tag{3.17}$$

The next lemma will be used to prove theorems 3.6.2 and 3.6.4.

Lemma 3.6.1 *Let L be an Archimedean lattice with Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ given by the formulas* (3.16) *and* (3.17)*. For $i = 1, \ldots, n_L$ consider the polynomials of two variables*

$$J_i^L(x,z) = b_0(z)\,a_i(x) - b_i(z)\,a_0(x).$$

If for any $i = 1, \ldots, n_L$ exists $\hat{\phi}_i : [0,1] \to [0,+\infty)$ such that $J_i^L(x, \hat{\phi}_i(x)) = 0$ one can define n_L functions $\phi_i : [0,+\infty) \to [0,+\infty)$ by $\phi_i(K) = \hat{\phi}_i(\tanh[K])$ such that

a)

$$\begin{aligned}f(K) &= H_0(\tanh[K])\\ &+ \frac{1}{\mathcal{S}\,8\,\pi^2}\int_B \ln\left[1 + \sum_{i=1}^{n_L} \frac{b_i(\phi_i(K))}{b_0(\phi_i(K))}\hat{\delta}_i(w)\right]dw.\end{aligned}$$

where

$$H_0(x) = \ln[2] - \frac{\mathcal{E}}{2\mathcal{S}}\ln[1-x^2] + \frac{\ln[a_0(x)]}{2\mathcal{S}}.\tag{3.18}$$

b) If $n_L = 1$ then

$$f(K) = H_0(\tanh[K]) - \frac{H_1(\phi(K))}{2} + \frac{1}{2}T(\phi(K)),$$

where

$$H_1(z) = \ln\left[\frac{q}{z}\right] + \frac{\ln[b_0(z)]}{\mathcal{S}}.\tag{3.19}$$

Proof. The proof is straightforward using the fact that for each Archimedean lattice L and each $i = 1, ..., n_L$, it reads

$$J_i^L(x, \hat{\phi}_i[x]) = 0 \Leftrightarrow \frac{a_i(x)}{a_0(x)} = \frac{b_i(\hat{\phi}_i(x))}{b_0(\phi_i(x))}.$$

For the next theorems we define the following subsets of the Archimedean lattices $\mathcal{L}(\mathcal{A})$

$$\mathcal{L}_1(\mathcal{A}) = \{Square, Triangle, Honeycomb, Kagome, Star\}$$
$$\mathcal{L}_2(\mathcal{A}) = \mathcal{L}(\mathcal{A}) - \mathcal{L}_1(\mathcal{A})$$

Note that for all Archimedean lattice $L \in \mathcal{L}_1(\mathcal{A})$ we have that $n_L = 1$.

We also recall that $K_c = arctanh[v_c]$ is the critical point of the Ising Free Energy $f(K)$ as is given in Table 2. The following theorem is a generalization of theorem 1 in [27] where it was proved in the case of the square lattice.
Our next theorem relates the Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ for all the Archimedean lattices L in $\mathcal{L}_1(A)$. The relation is given by means of an auxiliary function $\phi_L(K)$.

Theorem 3.6.2 *Let L be an Archimedean lattice in $\mathcal{L}_1(A)$ with Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ given by the formulas* (3.16) *and* (3.17)*. Then there exist a continuous function $\phi_L : [0, +\infty) \to [0, 1]$ such that*

a)

$$f_L(K) = H_2(K) + \frac{1}{2}T(\phi_L(K)),$$

where

$$H_2(K) = \frac{1}{2\mathcal{S}} \log \left[\frac{a_0(\tanh[K])(\cosh[K])^{2\mathcal{E}}}{b_0(\phi(K))}\right] + \log\left[\frac{2}{\sqrt{q}}\right] + \frac{\log[\phi(K)]}{2}.$$

b) $\phi(K_c) = 1$.

c) ϕ_L *is real analytic in* $(0, +\infty)$.

Furthermore the function $\phi_L : [0, \infty) \to [0, 1]$ is given by the following explicit expressions:

- *Square, see Fig. 19.*

$$\hat{\phi}_{sq}(x) = -\frac{4x\,(x^2-1)}{(x^2+1)^2},$$

$$\phi_{sq}(K) = \hat{\phi}_{sq}(\tanh(K)) = 2\tanh(2K)\textit{sech}(2K).$$

- *Triangle, see Fig. 20.*

$$\hat{\phi}_{tr}(x) = \frac{6(x-1)^2 x}{x^4 - 2x^3 + 6x^2 - 2x + 1},$$

$$\phi_{tr}(K) = \hat{\phi}_{tr}(\tanh(K)) = \frac{3\sinh(2K)}{\sinh^3(2K) + \cosh^3(2K)}.$$

- *Hexagonal, see Fig. 21.*

$$\hat{\phi}_{hx}(x) = \frac{3\sqrt{x^2 - x^4}}{\sqrt{3x^2+1}},$$

$$\phi_{hx}(K) = \hat{\phi}_{hx}(\tanh(K)) = \sqrt{\frac{9\sinh^2(2K)}{3\sinh^2(2K) + 2\left(\cosh^3(2K)+1\right)}}.$$

- *Kagome, see Fig. 22.*

$$\hat{\phi}(x) = \frac{k + \sqrt{l}}{t},$$

and

$$\begin{aligned} k =& - x^2 + 2x^3 - 2x^4 + 2x^5 - x^6, \\ l =& x^{14} - 6x^{13} + 23x^{12} - 58x^{11} + 107x^{10} - 152x^9 + 170x^8 - 152x^7 + 107x^6 \\ & - 58x^5 + 23x^4 - 6x^3 + x^2, \\ t =& x^8 - 4x^7 + 12x^6 - 20x^5 + 26x^4 - 20x^3 + 12x^2 - 4x + 1 \end{aligned}$$

- *Star, see Fig. 23.*

$$\begin{aligned} \hat{\phi}_{st}(x) =& - \frac{9k}{2} - \frac{1}{2}\sqrt{81k^2 - 9k} \\ & + \frac{1}{2}\sqrt{162k^2 - p + 99k}, \end{aligned}$$

and

$$\begin{aligned} p =& \frac{-5832k^3 - 3240k^2 + 432k}{\sqrt[4]{81k^2 - 9k}}, \\ k =& \frac{2x^7 - 3x^6 + 2x^5 - x^4}{(2x^4 + 2x^3 - 3x^2 + 2x - 1)^2}, \end{aligned}$$

$$\phi_{st}(K) = \hat{\phi}_{st}(\tanh[K]).$$

Proof. The proof is a direct consequence of Lemma 3.6.1. Using the fact that $\forall L \in \mathcal{L}_1(A)$

$$J^L(x, \hat{\phi}_L(x)) = 0$$

Observation1: Note that the function ϕ_{sq} given by this theorem is the same function given in Eq. 7 in [111].

Corollary 3.6.2.1 *Let L be a lattice in $\mathcal{L}_1(A)$ then we have the following relation between the spanning tree constant λ_L and the Ising critical energy $f_L^c = f_L(K_c)$*

$$\lambda_{sq} = 2f_{sq}^c - \ln[2], \tag{3.20}$$

$$\lambda_{tri} = 2f_{tri}^c - \ln\big(\frac{2}{\sqrt{3}}\big), \tag{3.21}$$

$$\lambda_{hc} = 2f_{hc}^c - \ln[2\sqrt{3}], \tag{3.22}$$

$$\lambda_{kg} = 2f_{hc}^c + \frac{1}{2}\ln[3] - \frac{4}{3}\ln[2] - \frac{1}{3}\ln[2+\sqrt{3}], \tag{3.23}$$

$$\lambda_{str} = 2f_{st}^c + \frac{1}{6}\ln\left[\frac{27}{640}(a\sqrt{3} + \sqrt{b\sqrt{3}+c} + d)\right] + 2\ln\left[\frac{2}{\sqrt{3}}\right], \tag{3.24}$$

where

$$\begin{aligned} a &= 22968, \\ b &= 1828939440, \\ c &= 3167815977, \\ d &= 39789. \end{aligned}$$

Proof. The proof is an immediate consequence of Theorem 3.6.2.
The relations (3.20),(3.21)(3.22) and (3.23) are already reported, see for example Ref. [25],and to the best of our knowledge the relation (3.24) is new.

Corollary 3.6.2.2 *If L is the linear, square, triangle or hexagonal lattice, we have that there exist $\phi_L(K)$ such that*

$$f_L(K) = \ln\sqrt{2\sinh[2K]} + \frac{1}{d}T(\phi_L(K))$$

where $d = 1$ if L is the linear lattice and $d = 2$ otherwise.

Proof. If L is the square, triangle or hexagonal, the proof is a direct consequence of theorem 1, now we prove the case when L is the linear lattice. It is known that

$$f_L(K) = \ln\left(e^K + \sqrt{e^{2K} - 2sinh(2K)}\right),$$

Now, using this following identity

$$\frac{1}{2\pi}\int_0^{2\pi} \ln(1 \pm r sin(\theta))d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln(1 \pm r cos(\theta))d\theta = \ln\left(\frac{1}{2}(1+\sqrt{1-r^2})\right)$$

We have

$$f_L(K) = \frac{1}{2\pi}\int_0^{2\pi} ln(2e^K - 2\sqrt{2sinh(2K)}cos(\theta))d\theta,$$

so,

$$f_L(K) = \ln(2) + \frac{1}{2\pi}\int_0^{2\pi} \ln\left(\frac{e^K}{\sqrt{2sinh(2K)}} - cos(\theta)\right)d\theta + \ln(\sqrt{2sinh(2K)})$$

Now, since the Spanning tree generating function is given by

$$T(z) = \ln(2) + \frac{1}{2\pi}\int_0^{2\pi} ln\left(\frac{1}{z} - cos(\theta)\right)d\theta$$

The result follows choosing the function

$$\phi_L : (0, +\infty) \to (0, 1]$$
$$\phi_L(K) = \frac{\sqrt{2sinh(2K)}}{e^K}$$

Lemma 3.6.3 *Let L be an Archimedean lattice in $\mathcal{L}_2(A)$ with Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ given by the formulas* (3.16) *and* (3.17)*. Then there exists n_L continuous functions $\hat{\phi}_i : [0,1] \to [0,+\infty)$ such that $J_i^L(x, \hat{\phi}_i(x)) = 0$ for all $i = 1, ..., n_L$ with the following properties*

1. *$\hat{\phi}_i$ are real analytic in $(0,1)$, for all $i = 1, ..., n_L$.*
2. *$\hat{\phi}_i(0) = \hat{\phi}_i(1) = 0$ for all $i = 1, ..., n_L$.*
3. *If $\hat{\phi}_i \neq \hat{\phi}_1$,, then $0 < \hat{\phi}_i(x) < 1$ for all $x \in (0,1)$*
4. *There exists two points $x_a < x_b \in (0,1)$ (see Table 5) such that*
   - $x_c \in (x_a, x_b)$,
   - $\hat{\phi}_1(x_a) = \hat{\phi}_1(x_b) = 1$,
   - $0 < \hat{\phi}_1(x) < 1$ *if* $x \in (0, x_a) \cup (x_b, 1)$,
   - $1 < \hat{\phi}_1(x) < 1.05$ *if* $x \in (x_a, x_b)$.
5. *If L is the bridge lattice, then*

$$\hat{\phi}_1(x) = \frac{\sqrt{q_1}}{q_2},$$
$$\hat{\phi}_2(x) = \frac{5p_1 + \sqrt{p_2}}{p_3},$$

and

$$\begin{aligned}
q_1 =& -x^8 + 3x^6 - 3x^4 + x^2 \\
q_2 =& 1 + 5x^2 + 8x^3 + 7x^4 + 16x^5 + 15x^6 + 8x^7 + 4x^8, \\
p_1 =& -5x^8 - 8x^7 - 12x^6 - 16x^5 - 10x^4 - 8x^3 - 4x^2 - 1 \\
p_2 =& 21x^{16} + 66x^{15} + 145x^{14} + 286x^{13} + 430x^{12} + 538x^{11} \\
& + 587x^{10} + 582x^9 + 534x^8 + 390x^7 + 247x^6 + 154x^5 \\
& + 70x^4 + 30x^3 + 13x^2 + 2x + 1, \\
p_3 =& 4x^8 + 6x^7 + 7x^6 + 14x^5 + 15x^4 + 10x^3 + 5x^2 + 2x + 1.
\end{aligned}$$

Proof. We prove only for the case of the cross lattice the other cases are proven in a similar way.

We have that for all $x \in (0, 1)$,

$$\begin{aligned}
a_0[x] >& 0, \\
a_1[x] <& 0, \\
a_2[x] <& 0, \\
a_3[x] >& 0,
\end{aligned}$$

and for all $z \in (0, 1.05)$, (the symbol $'$ means first derivative)

$$\begin{aligned}
b_0'[z] <& 0, \\
b_1'[z] <& 0, \\
b_2'[z] <& 0, \\
b_3'[z] >& 0,
\end{aligned}$$

so for all $x, z \in (0, 1) \times (0, 1.05)$ and $i = 1, 2, 3$, we have

$$\partial_z(J_i(x, z)) = a_0[x]b_i'[z] - a_i[x]b_0'[z] \neq 0 \tag{3.25}$$

We also have for $i = 1, 2, 3$ and $y \in (0, 1.05)$

$$J_i(0, y) = 0 \Rightarrow y = 0 \tag{3.26}$$

$$J_i(1, y) = 0 \Rightarrow y = 0 \tag{3.27}$$

(i) We first construct $\hat{\phi}_i$, for $i = 2, 3$
We have for $x \in (0, 1)$.

$$J_i(x, 0)J_i(x, 1) < 0, \tag{3.28}$$

by eqs. (3.25), (3.28) and Theorem B.1.5(b) we have that there exists a real analytic function $f_i : (0,1) \to (0,1)$ such that

$$J_i(x, f_i(x)) = 0, \quad \forall x \in (0,1)$$

we define $\hat{\phi}_i$ as

$$\hat{\phi}_i(x) = \begin{cases} 0 & x = 0, \\ \\ f_i(x) & x \in (0,1), \\ \\ 0 & x = 1, \end{cases}$$

By Eqs. (3.26),(3.27) and Theorem B.1.6, $\hat{\phi}_i$ is continuous in $[0,1]$.

(ii) Finally we need to prove the existence of the function $\hat{\phi}_1$ and the points x_a and x_b.

The polynomial $J_1(x,1)$ have two real roots x_a, x_b in $(0,1)$ such that

$$J_1(x,1) < 0 \qquad \forall x \in (0,x_a) \cup (x_b,1) \tag{3.29}$$

$$J_1(x,1) > 0 \qquad \forall x \in (x_a,x_b) \tag{3.30}$$

We also have for all $x \in (0,1)$

$$J_1(x,0) > 0. \tag{3.31}$$

by eqs. (3.25),(3.31) and (3.29) and Theorem B.1.5(b), for $\alpha \in \{a,b\}$ there exist a function

$$f_\alpha : I_\alpha \to (0,1)$$

such that

$$J_1(x, f_\alpha(x)) = 0, \quad \forall x \in I_\alpha$$

where $I_a = (0, x_a)$ and $I_b = (x_b, 1)$.

Now we note that

$$J_1(x, 1.05) < 0, \forall \in (0,1),$$

so by eqs (3.25) and (3.30) and Theorem B.1.5(b) there exists a function $f_{ab} : (x_a, x_b) \to (1, 1.05)$ such that

$$J_1(x, f_{ab}(x)) = 0 \quad \forall x \in (x_a, x_b)$$

Now we define $\hat{\phi}_1 : [0,1] \to [0, 1.05)$ as

$$\hat{\phi}_1(x) = \begin{cases} 0 & x = 0, \\ f_a(x) & x \in (0, x_a) \\ 1 & x = x_a, \\ f_{ab}(x) & x \in (x_a, x_b), \\ 1 & x = x_b, \\ f_b(x) & x \in (x_b, 1), \\ 0 & x = 1, \end{cases}$$

Since $J_1(x, \phi_1(x)) = 0, \forall x \in (0,1)$, by Theorem B.1.5(a) this function is real analytic in $(0,1)$ and by Eqs. (3.26),(3.27) and Theorem B.1.6, $\hat{\phi}_1$ is continuous in $[0,1]$.

Note that by Lemma 3.6.3 we have that for each $L \in \mathcal{L}_2(\mathcal{A})$,

$$x_a \sim x_b \sim x_c$$

Table 5 – Values for x_a and x_b given by Lemma 3.6.3

Lattice L	Label [1]	n_L	x_a	x_b	x_c [a]
Bridge	(3^3,4^2)	2	0.315861...	0.355582...	1/3
Bathroom	(4,8^2)	2	0.567257...	0.6446...	0.601232...
Puzzle	(3^2,4,3,4)	3	0.308764...	0.353893...	0.329024...
Ruby	(3,4,6,4)	3	0.429997...	0.441133...	0.435421...
Maple Leaf	(3^4,6)	3	0.337389...	0.351785...	0.344296...
Cross	(4,6,12)	3	0.608415...	0.625407...	0.616606...

[b] x_c coincides with $\tanh[K_c]$ for K_c the critical point of the corresponding Ising model on L.

Our next theorem relate the Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ for all the Archimedean lattices L. The relation is given by means of a set of auxiliary functions $\phi_1(K), ..., \phi_{n_L}(K)$

Theorem 3.6.4 *Let L be an Archimedean lattice with Ising free energy $f_L(K) = \hat{f}_L(\tanh[K])$ and STGF $T_L(z)$ given by the formulas* (3.16) *and* (3.17). *Then there exists n_L functions $\phi_i : [0, +\infty) \to [0, +\infty)$ for all $i = 1, ..., n_L$, such that*

$$f_L(K) = H_0(\tanh[K]) + \frac{1}{\mathcal{S}8\pi^2} \int_B \ln\left[1 + \sum_i^{n_L} \frac{b_i[\phi_i(K)]}{b_0[\phi_i(K)]} \delta_i[w]\right] dw.$$

furthermore

1. *ϕ_i are real analytic in $(0, +\infty)$, for all $i = 1, ..., n_L$.*

2. *$\phi_i(0) = 0, \phi_i(+\infty) = 0$ for all $i = 1, .., n_L$*

3. *If $n_L = 1$, then $\phi(K_c) = 1$,*

4. *If $n_L = 2$ or $n_L = 3$, then*

   a) *If $\phi_i \neq \phi_1$, then $0 < \phi_i(K) < 1$ for all $K \in (0, \infty)$,*

   b) *there exists two points $K_a < K_b \in (0, \infty)$ (see Table 6) such that*

      - $K_c \in (K_a, K_b)$,
      - $\phi_1(K_a) = \phi_1(K_b) = 1$,
      - $0 < \phi_1(K) < 1$ *if* $x \in [0, K_a) \cup (K_b, \infty)$,
      - $1 < \phi_1(K) < 1.05$ *if if* $K \in (K_a, K_b)$.

   c) *If L is the Bridge lattice*

$$\begin{aligned}\phi_1(K) =& \hat{\phi}_1(\tanh[K]),\\ \phi_2(K) =& \hat{\phi}_2(\tanh[K]),\end{aligned}$$

where $\hat{\phi}_1(x)$ and $\hat{\phi}_2(x)$ are given by Lemma 3.6.3.

Proof. The proof is an immediate consequence of Lemma 3.6.1 and Lemma 3.6.3, defining for all $i = 1, ..., n_L$,

$$\phi_i(K) = \hat{\phi}_i(\tanh[K]).$$

Note that by Theorem 3.6.4, we have that for each $L \in \mathcal{L}_2(\mathcal{A})$,

$$K_a \sim K_b \sim K_c$$

Table 6 – Values for K_a and K_b given by Theorem 3.6.4

Lattice L	Label [1]	n_L	K_a	K_b	K_c [a]
Bridge	(3^3,4^2)	2	0.327043...	0.371819...	0.346574...
Bathroom	(4,8^2)	2	0.643469...	0.766004...	0.695074...
Puzzle	(3^2,4,3,4)	3	0.319178...	0.369887...	0.341729...
Ruby	(3,4,6,4)	3	0.459893...	0.473636...	0.466566...
Maple Leaf	(3^4,6)	3	0.351143...	0.36748...	0.358958...
Cross	(4,6,12)	3	0.7064...	0.733837...	0.71951...

[b] K_c is the critical point of the corresponding Ising model on L.

3.7 PLOT OF THE AUXILIARY FUNCTIONS $\phi_i(K)$ AND $\hat{\phi}_i(x)$

In this section we present the plots of the auxiliary functions $\phi_i(K)$ and $\hat{\phi}_i(x)$ for all the Archimedean lattices L, whose existence was given in our Theorem 3.6.2 and Theorem 3.6.4. We recall that $\phi_i(K) = \hat{\phi}_i(\tanh[K])$ and $\hat{\phi}_i(x)$ are functions that are given by the implicit equations $\forall i = 1, .., n_L$

$$J_i(x, \phi_i(x)) = 0, \forall (x, z) \in \mathcal{I}$$

where $J_1(z, x), .., J_{n_L}(x, z)$ are polynomials in two variables and $\mathcal{I}$ is a square region in the plane $x - z$, more precisely

$$\mathcal{I} = [0, 1] \times [0, 1.05].$$

We remark that for each lattice with $n_L = 1$, it is indicated the point $(x_c, 1)$ in the plane $x - z$ and the point $(K_c, 1)$ in the plane $K - z$.

3.7.1 Square lattice, $n_L = 1$.

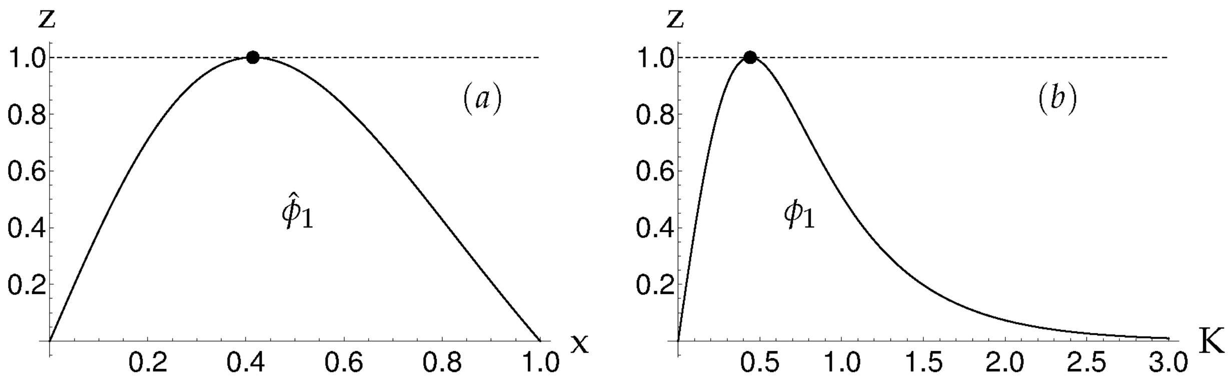


Figure 19 – (a) Graph of the function $z = \hat{\phi}_{sq}(x)$ (b) Graph of the function $z = \phi_{sq}(K) = \hat{\phi}_{sq}(\tanh(K))$.

3.7.2 Triangle lattice, $n_L = 1$.

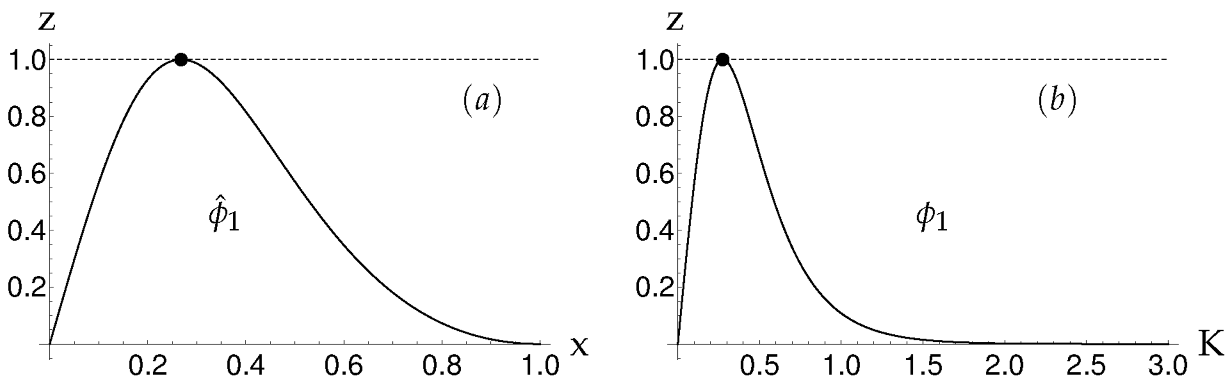


Figure 20 – (a) Graph of the function $z = \hat{\phi}_{tr}(x)$ (b) Graph of the function $z = \phi_{tr}(K) = \hat{\phi}_{tr}(\tanh(K))$.

3.7.3 Hexagonal lattice, $n_L = 1$.

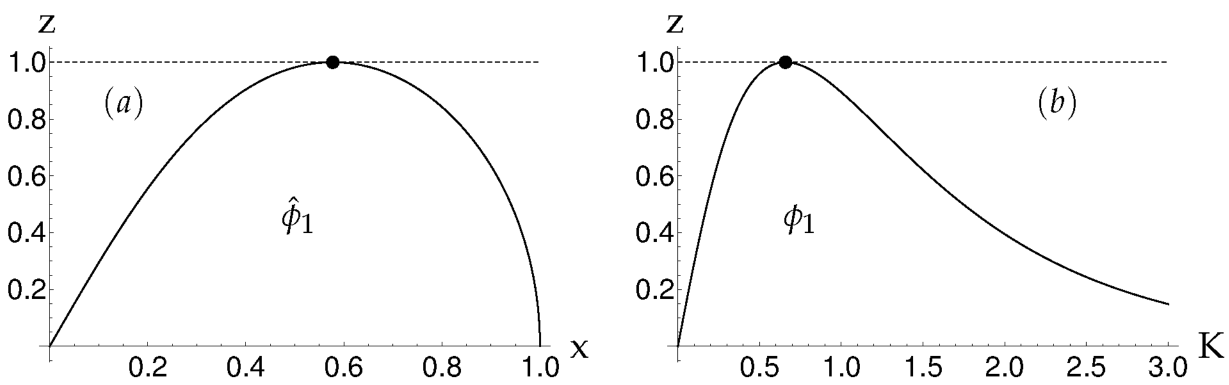


Figure 21 – (a) Graph of the function $z = \hat{\phi}_{hx}(x)$ (b) Graph of the function $z = \phi_{hx}(K) = \hat{\phi}_{hx}(\tanh(K))$.

3.7.4 Kagome lattice, $n_L = 1.$

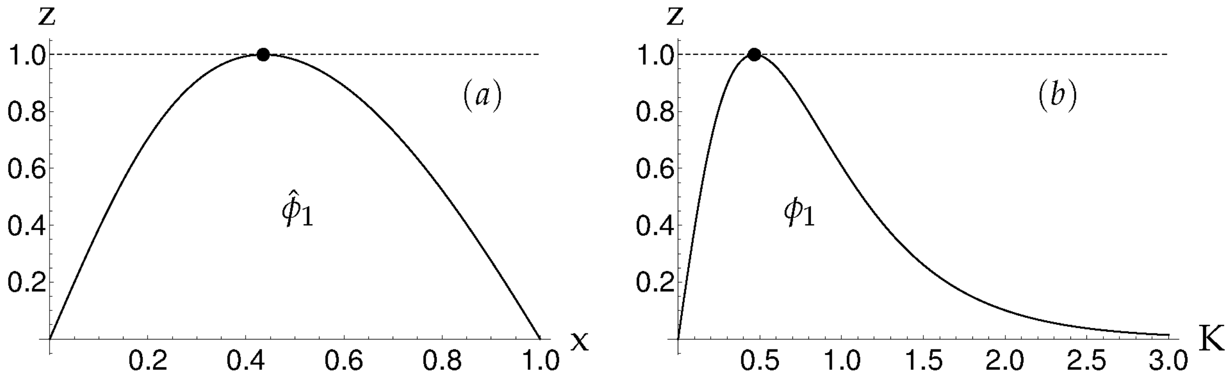


Figure 22 – (a) Graph of the function $z = \hat{\phi}_{kg}(x)$ (b) Graph of the function $z = \phi_{kg}(K) = \hat{\phi}_{kg}(\tanh(K))$.

3.7.5 Star lattice, $n_L = 1.$

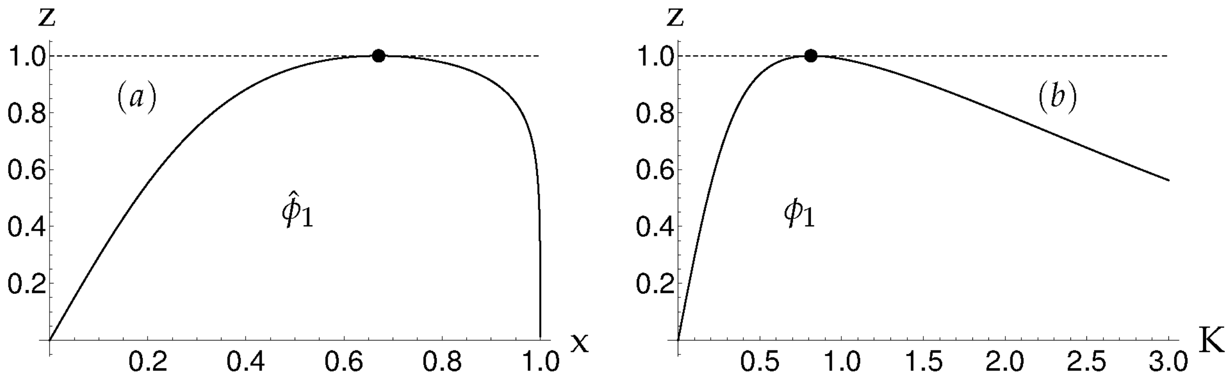


Figure 23 – (a) Graph of the functions $z = \hat{\phi}_{st}(x)$ (b) Graph of the function $z = \phi_{st}(K) = \hat{\phi}_{st}(\tanh(K))$.

3.7.6 Bathroom lattice, $n_L = 2$.

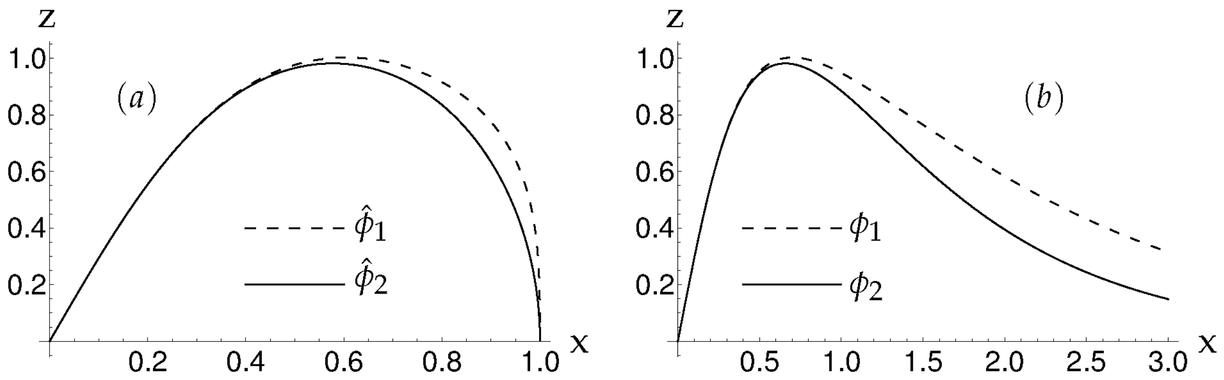


Figure 24 – (a) Graphs of the functions $z = \hat{\phi}_i^{br}(x)$, $i = 1, 2$. (b) Graphs of the functions $z = \phi_i^{br}(K) = \hat{\phi}_i^{br}(\tanh(K))$, $i = 1, 2$.

3.7.7 Bridge lattice, $n_L = 2$.

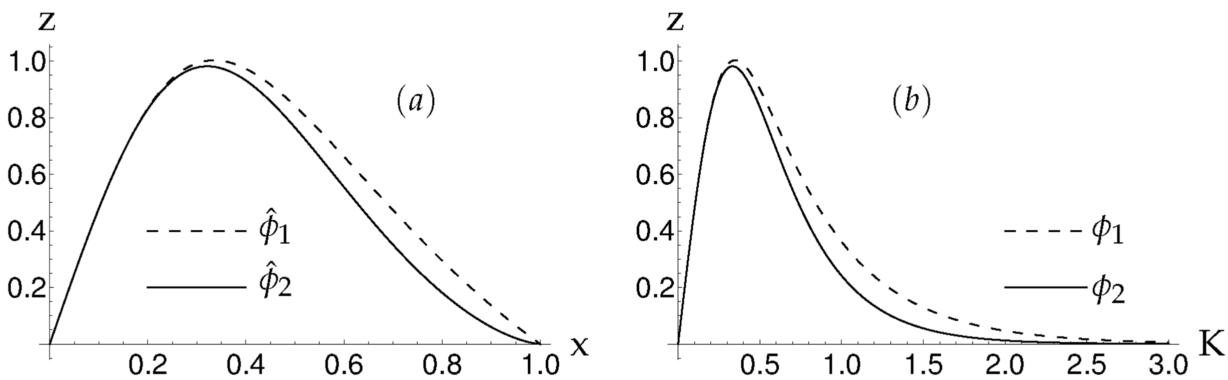


Figure 25 – (a) Graphs of the functions $z = \hat{\phi}_i^{brg}(x)$, $i = 1, 2$ (b) Graphs of the functions $z = \phi_i^{brg}(K) = \hat{\phi}_i^{brg}(\tanh(K))$, $i = 1, 2$.

3.7.8 Puzzle lattice, $n_L = 3$.

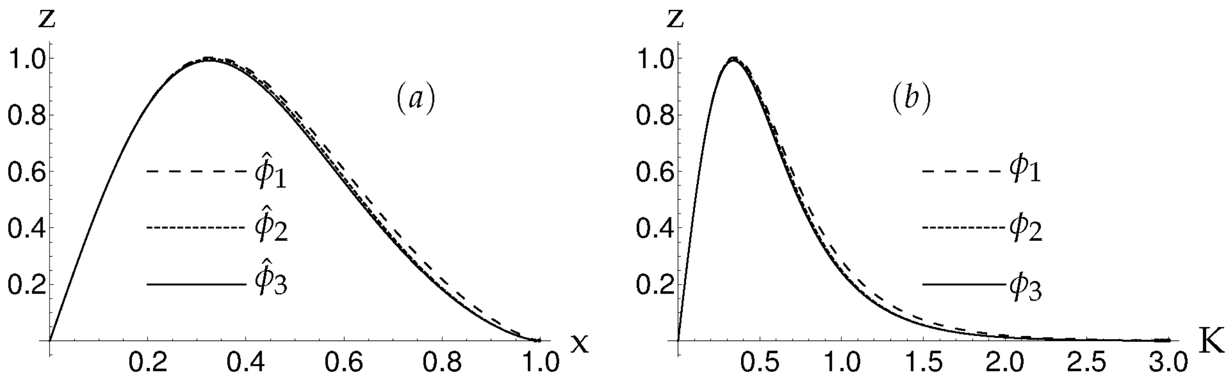


Figure 26 – (a) Graphs of the functions $z = \hat{\phi}_i^{pz}(x)$, $i = 1, 2, 3$. (b) Graphs of the function $z = \phi_i^{pz}(K) = \hat{\phi}_i^{pz}(\tanh(K))$, $i = 1, 2, 3$.

3.7.9 Ruby lattice, $n_L = 3$.

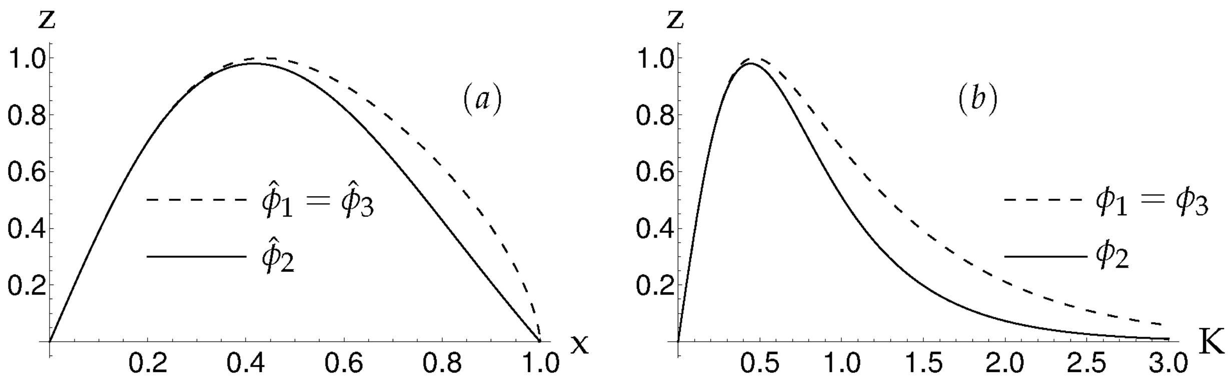


Figure 27 – (a) Graphs of the functions $z = \hat{\phi}_i^{rb}(x)$, $i = 1, 2, 3$. (b) Graphs of the function $z = \phi_i^{rb}(K) = \hat{\phi}_i^{rb}(\tanh(K))$, $i = 1, 2, 3$.

3.7.10 Maple lattice, $n_L = 3$.

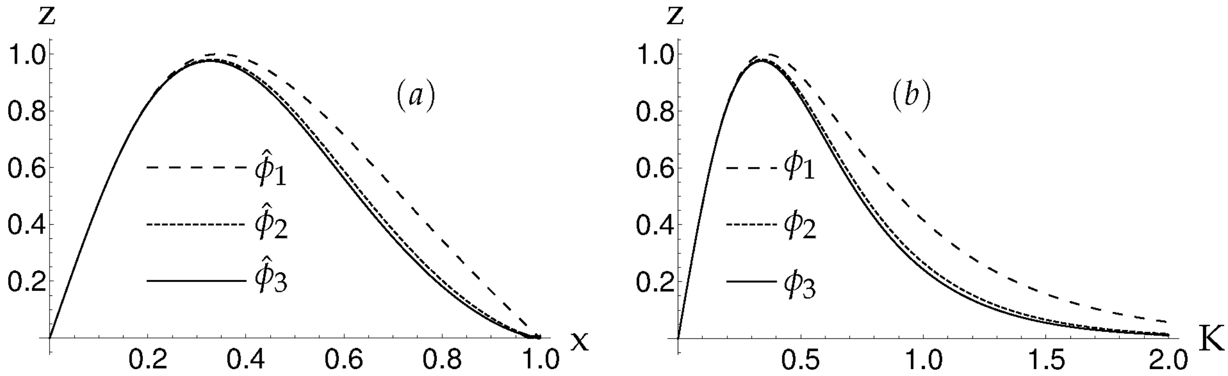


Figure 28 – (a) Graphs of the functions $z = \hat{\phi}_i^{mp}(x)$, $i = 1, 2, 3$. (b) Graphs of the function $z = \phi_i^{mp}(K) = \hat{\phi}_i^{mp}(\tanh(K))$, $i = 1, 2, 3$.

3.7.11 Cross lattice, $n_L = 3$.

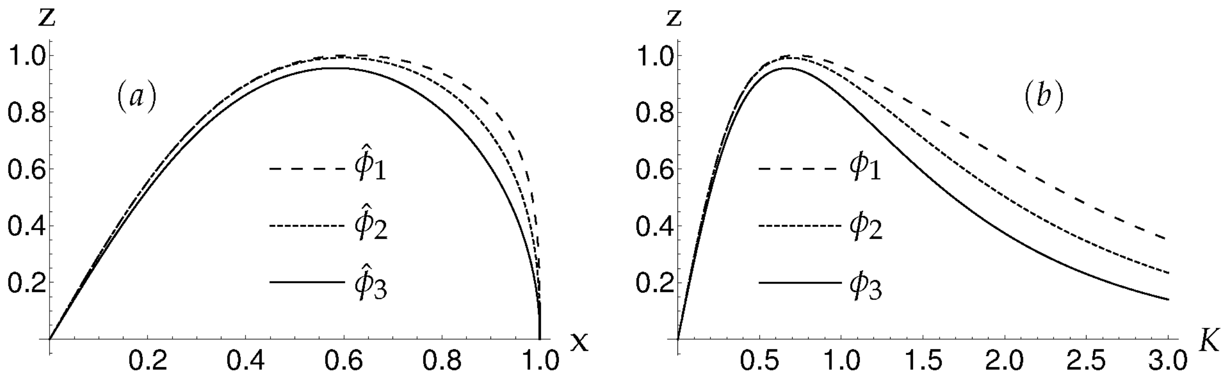


Figure 29 – (a) Graphs of the functions $z = \hat{\phi}_i^{cr}(x)$, $i = 1, 2, 3$. (b) Graphs of the function $z = \phi_i^{cr}(K) = \hat{\phi}_i^{cr}(\tanh(K))$, $i = 1, 2, 3$.

3.7.12 eSTGF and Ising free energy of the Martini lattice

In the previous sections we worked with the STFG of the eleven Archimedean lattices, which are vertex-transitive, now we will work with a non-vertex transitive lattice L, called the martini lattice (fig.7) , so we can not use the STGF but we can use our formula for the eSTGF . Actually the eSTGF was calculated in (sec 2.5.2), and we write it again here.
Since by Theorem 2.4.2, the eSTGF have a closed form expression for periodic lattices in any dimension (that don't need to be vertex transitive), we think that a deeper study of the eSTGF could be useful in determine properties of the Ising free energy in any dimensions, particularly in the important case of the simple cube lattice in 3D. We hope to find more connections between eSTGF and $f(K)$ in future projects.

$$T_L(z) = \frac{1}{4}\ln\left[\frac{6\,(z+3)}{z}\right] + \frac{1}{4}\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln\left[g(z) - \frac{1}{3}\,(\Delta_0+\Delta_+)\right]dw_1\,dw_2, \quad (3.32)$$

where

$$g(z) = \frac{z^3-5z^2-3z+9}{2z^3}.$$

Using lemma 4.3 in Ref.[108] we calculated the The Ising free energy of the martini lattice and is given by the following formula.

$$\begin{aligned} f_L(K) &= ln[2] + \frac{3}{2}\ln[\cosh(K)] + \frac{1}{8(2\pi)^d} \\ &\times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} ln\Big[a_0 + a_1(\Delta_0+\Delta_+)\Big]\,dw_1\,dw_2, \end{aligned} \quad (3.33)$$

where

$$\begin{aligned} a_0 &= \hat{a}_0(\tanh[K]) \\ a_1 &= \hat{a}_1(\tanh[K]) \end{aligned}$$

and

$$\begin{aligned} \hat{a}_0(x) &= 3(x+1)^2(x^6+x^4-2x^3+3x^2-2x+1) \\ \hat{a}_1(x) &= 2(x+1)^2(x-1)x^3(x^2+1) \end{aligned}$$

Also by theorem 1.1 and proposition 2.2 in ref [108] we calculated the critical inverse temperature K_c of $f(K)$ and is given by

$$K_c = arctanh\left(\frac{1}{9}\left(\sqrt[3]{27\sqrt{17}+109} - \frac{8}{\sqrt[3]{27\sqrt{17}+109}} + 1\right)\right)$$

Before proving our next theorem we remark that, to the best of our knowledge, the $f_L(K)$ and K_c that are reported above in this thesis, are not found in the literature.

Theorem 3.7.1 *Let L be an Martini lattice with Ising free energy $f_L(K)$ and $eSTGF$ $T_L(z)$ given by the formulas* (3.33) *and* (3.32). *Then we found the following relation between $T_L(z)$ and $f_L(K)$*

$$f_L(K) = C[K] + \frac{1}{2}T_L[\phi(K)],$$

where

$$C[K] = \frac{1}{8}\log\Big(R(\tanh[K], \phi[K])\Big) + \log\left(\frac{2}{\sqrt{3}}\right) + \frac{\log(\phi[K])}{2},$$

and

$$R(x, y) = \frac{-81\hat{a}_0(x)}{(1-x^2)^6(-3y^3+15y^2+9y-27)(3+y)}.$$

Furthermore, the function $\phi : [0,\infty) \to [0,1]$ satisfies that $\phi[K_c] = 1$ and its given by the following explicit expressions:

$$\phi[K] = \hat{\phi}[\tanh[K]],$$

where

$$\begin{aligned}
\hat{\phi}[x] &= \frac{A}{768\sqrt[3]{2}C} + \frac{B}{384\ 2^{4/3}C} + \frac{5\left(x^6 - x^5 + x^4 - x^3\right)}{C},\\
A &= \sqrt[3]{L+\sqrt{n}},\\
B &= \sqrt[3]{L-\sqrt{n}},\\
C &= 6x^6 - 3x^5 + 4x^4 - 5x^3 + 3x^2 - 2x + 1,\\
n &= L^2 + 4S^3,
\end{aligned}$$

and

$$\begin{aligned}
L = {} & -204749144064x^{18} + 235099127808x^{17} - 327508033536x^{16}\\
& +534975086592x^{15} - 687630974976x^{14} + 1148316549120x^{13}\\
& -1611267047424x^{12} + 1917484793856x^{11} - 1951458656256x^{10}\\
& +1709111771136x^{9} - 1276058271744x^{8} + 811295834112x^{7}\\
& -432147529728x^{6} + 183458856960x^{5} - 61152952320x^{4}\\
& +12230590464x^{3},\\
S = {} & -25362432x^{12} + 45416448x^{11} - 67239936x^{10} + 90832896x^{9}\\
& -70778880x^{8} + 54263808x^{7} - 34209792x^{6},\\
& +10616832x^{5} - 5308416x^{4} + 1769472x^{3}.
\end{aligned}$$

Note that in Theorem 3.7.1 all the lattices are vertex-transitive and in Theorem 3.6.2, the martini lattice is non-vertex transitive, so one important conclusion of Theorem 3.7.1 is that it supports the idea that similar relations could be obtain even is the lattice is non-vertex transitive. We hope to explore this idea in the future.

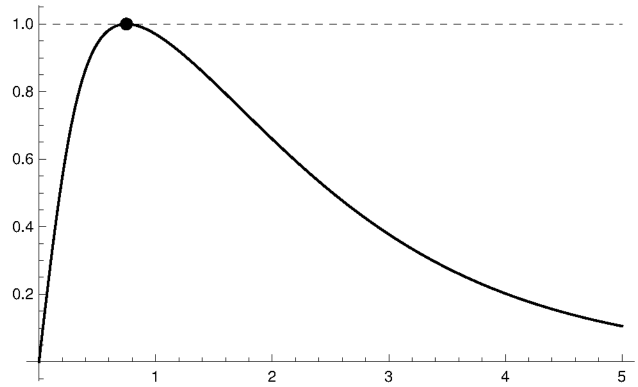


Figure 30 – The plot of the function $\phi[K]$ for the martini lattice in Theorem 3.7.1.

3.8 CONCLUSION.

In this chapter, we calculated the formulas for the isotropic Ising free energy for all the Archimedean lattices, our results match with the Ising free energy that are already published in the literature, namely, the Square, Triangle, Hexagonal, Kagome, Bathroom and Star lattices,[28, 32, 33], the isotropic Ising free energy for the other 5 Archimedean lattices that we present in this chapter, to the best of our knowledge are not reported anywhere. With the $STGF$ calculated for all the Archimedean lattices in Chapter 2 we show some relations between the $STGF$ and the Ising free energy for the eleven Archimedean lattices. Theorem 3.6.2 and Theorem 3.6.4 are our main results in this chapter, which say essentially that we can reconstruct the Ising free energy for all the Archimedean lattices using its $STGF$ via a set of auxiliary real analytic functions. These new contributions extend the relation between the $STGF$ and the Ising free energy of the square lattice given by Theorem 1 in [27] considering all the Archimedean lattices. Furthermore at the end of this chapter, we found also a connection but in the case of an important non-vertex transitive lattice called the martini lattice, this new relation required our notion of the extended $STGF$ ($eSTGF$) and is given in Theorem 3.7.1.

4 CONNECTIONS BETWEEN THE WEIGHTED SPANNING TREES AND THE ANISOTROPIC ISING MODEL AND DIMERS MODEL.

4.1 ABSTRACT

We found connections between the $eSTGF$ and the isotropic Ising model, for the eleven Archimedean lattices and the martini lattice in chapter 3. It is natural to ask whether similar relations can be found between spanning trees and the anisotropic Ising model and also with some other systems defined on weighted graphs like the Dimer model. We define the notion of a weighted Spanning tree generating function $(wSTGF)$ and could find some relations with the anisotropic Ising model with arbitrary couplings that generalize the results on chapter 2, and with Dimer model on the square and triangle lattice.

4.2 INTRODUCTION

Finding a formula for the function $STGF$, originally defined in [26], provides us with a tool to extend the work developed in [27], consisting in finding relations between spanning trees and Ising models. Futhermore since we define a extended spanning tree generating function $eSTGF$ in chapter 2, we can also investigate relations when the lattice is regular but non-vertex-transitive (like the martini lattice). We found such relations between the $STGF$ and all the Archimedean lattices and between the $eSTF$ and the martini lattice in chapter 3.

Given a graph $G = (V, E)$ a weight system is simply an assigment $w : E \to (0, \infty)$ and a pair (G, w) is called a weighted graph. Weighted graphs are really important in physics, for example, the anisotropic Ising model can be defined on a weighted graph, where the positive weights on every edge $J = (J_e)_{e \in E}$ are called couplings, and the energy of a spin configuration σ is given by the hamiltonian

$$H(\sigma) = \sum_{(i,j)\in E} \sigma_i \sigma_j J_{(i,j)}$$

Onsager[28] proved that the Ising model in the thermodynamic limit for the square lattice with the coupling constant for every horizontal edge equal to J_1, and for every vertical edge equal to J_2, the following expression

$$f(K_1, K_2) = \ln[2] + \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \ln[\cosh[2K_1] \cosh[2K_2] + \sin[2K_1] \cos[\theta_1] + \sin[2K_2] \cos[\theta_2]] \, d\theta$$

Where $K_i = J_i(k_B T)^{-1}$, for $i = 1, 2$. Another important example where a weighted lattice is used in Physics is the Dimer model. The dimer model arose initially as an attempt to describe the adsorption of diatomic molecules on the surface of crystals, and has had been successful in describing the behavior of partially dissolved crystals in equilibrium. Futhermore, quantum versions of the dimer model were also proposed in studies of high temperature superconductors, specifically in the study of $SU(2)$ singlet dominated phases in various spin models [115]. In this chapter we define a weighted Spanning tree generating function that allows a positive weight in every edge and also that allows a non-regular lattice. So if the weights are equal to 1 and the lattice is regular we have $eSTGF = wSTGF$. We found that if we choose special weights the $wSTGF$ is related to the Dimer model in the case of the square lattice and triangle lattice. We also calculated the $wSTGF$ for two non-regular lattices, the Union jack lattice (Fig.34) and the Cairo pentagonal lattice (Fig. 35). The Cairo pentagonal is the dual of the Archimedean lattice $(3^2, 4, 4)$ (Fig. 6) and is not regular since the degrees in it are 3 or 4. The Union Jack lattice is the dual of the Archimedean lattice $(4.8.8)$ (Fig. 6) and is not regular since the degrees in it are 4 or 6.

4.3 BASIC CONCEPTS

4.3.1 Dimer Configuration

A perfect matching or dimer configuration (also known as close-packed dimers) on a graph $G = (V(G), E(G))$ is a subset $D \subset E(G)$, where the edges satisfy the condition that each vertex of G is incident to exactly one edge in D, see Fig. 31, every edge e in a dimer configuration D is called a dimer. Given a finite graph G, the set $\mathcal{D}(G)$ is the set of all dimer configurations of G. A weight on G is a positive real-valued function $w : E(G) \to (0, \infty)$. The pair (G, w) is called a weighted graph, $w(e)$ is called a dimer weight and the Dimer partition function is defined as

$$Z(G, w) = \sum_{D \in \mathcal{D}(G)} W(D),$$

where

$$W(D) = \prod_{e \in D} w(e)$$

If (G_n, w_n) is a sequence of finite graph such that the limit

$$F_{DM}(w) = \lim_{n \to \infty} \frac{Z(G_n, w_n)}{\frac{|V(G_n)|}{2}}$$

exist, then f_w is called the Dimer entropy or the free energy per dimer of the sequence (G_n, w_n). The Dimer entropy have been calculated for many periodic lattices, such as triangle [116], square [117] and the Cairo pentagonal lattice [118], which is a non-regular lattice

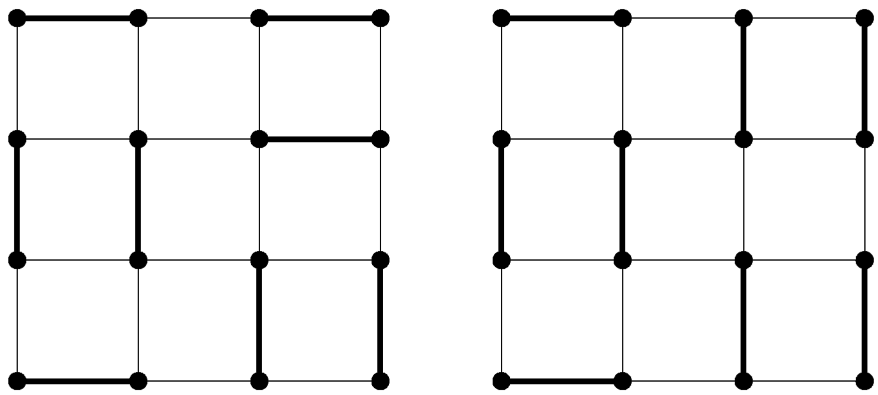


Figure 31 – Two different Dimer configuration for the Grid graph 4x4

4.3.2 Weighted periodic lattices

In Chapter 2 and chapter 3, we worked with $d-$periodic lattices, for convenience we repeat here its definition.
L is an infinite d–periodic — or just periodic for short — graph (lattice) if [23, 65].:

1. its vertices are labeled in $Z^d \times S$, where $S = \{1, 2, \ldots, \mathcal{S}\}$, with $|S| = \mathcal{S}$ finite (thus, for $v_x \in V(L)$ we write $x = (k, s)$, with $k \equiv (k_1, k_2, \ldots, k_d)$, all the k_n's being integers and $s \in S$);
2. the adjacency matrix A_L of L has the following property

$$A_L((k,s),(l,t)) = A_L((k-l,s),(0,t)), \quad \forall\, k,l \in Z^d, \quad \forall\, s,t \in S. \tag{4.1}$$

Note that a $d-$periodic lattice does not need to be q–regular nor vertex-transitive[66].
Now we will endow a $d-$periodic lattice with an extra structure defined by a positive weight in every edge, so that we get the following definition.

A weighted $d-$periodic $L = (Z^d, Z^d \times \mathcal{S}, w)$ is a $d-$periodic graph $L = (Z^d, Z^d \times \mathcal{S})$ with a function $w : V(L) \times V(L) \to (0, \infty)$
such that $w(k, l) = 0$ if $(k, l) \notin E(G)$ and

$$w((k,i),(l,j)) = w((k-l),i),(0,j)) \quad \forall \quad k,l \in Z^d. \tag{4.2}$$

The function $\Lambda : R^d \to C^{\mathcal{S}\times\mathcal{S}}$, defined by (for all θ_n's in $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_d)$ reals)

$$\Lambda(\theta) = \sum_{k \in Z^d} \mu(k) \exp[i\, k \cdot \theta]$$

is called the weighted structure function of L, where $[\mu(k)]_{ij} = p((k,i),(0,j))$ with

$$p((k,i),(0,j)) = \frac{w((k,i),(l,j))}{\delta(i)} \tag{4.3}$$

where

$$\delta(i) = \sum_{(l,j)\in Z^d \times \mathcal{S}} w((0,i),(l,j))$$

Note that if for some constant $c > 0$, $w((k,i),(l,j) = c$, for all $((k,i),(l,j)) \in E(G)$ then w is simply the adjacency matrix of L and $\delta(i) = Deg((0,i))$.
Note that every graph can be consider weighted if every edge have a weight equal to 1, and the concept of $p_n(0,s)$ and lattice green function $P(0,s,z) = \sum p_n(0,s)z^n$ given in chapter 1, extend naturally if we use the weighted transfer matrix of Eq. (4.3).

Theorem 4.3.1 *For $L = (Z^d, E(L), w)$ an infinite weighted d–periodic lattice, its weighted spanning tree constant is given by*

$$z_L = \frac{1}{\mathcal{S}}\sum_{i=1}^{\mathcal{S}} \ln[\delta(i)] - \frac{1}{\mathcal{S}}\sum_{n=1}^{\infty}\sum_{n=1}^{\mathcal{S}} \frac{p_n(0,s)}{n}. \tag{4.4}$$

Proof. The proof is an immediate consequence of theorem 5.1 and remark 5 in [23].

We denote $B = [0,2\pi]^d$.

4.4 EXTENSION OF STGF FOR WEIGHTED PERIODIC GRAPHS

The following theorem is an extension of Theorem 3 of chapter 1, given that now we are considering lattices that can be non-regular and weighted.

Theorem 4.4.1 *Let $L = (Z^d, E(L), w)$ be a weighted $d-$periodic for L, we define weighted spanning tree generating function $wSTGF$, $T_w : (-1,0) \cup (0,1) \to R$, as the well behaved series*

$$T_w(z) = \frac{1}{\mathcal{S}}\sum_{i=1}^{\mathcal{S}} \ln[\delta(i)] - \ln[|z|] - \frac{1}{\mathcal{S}}\sum_{n=1}^{\infty}\sum_{n=1}^{\mathcal{S}} \frac{p_n(0,s)}{n}. \tag{4.5}$$

Then:
(i) $T_w(z)$ can be cast as

$$T_w(z) = \frac{1}{\mathcal{S}}\sum_{i=1}^{\mathcal{S}} \ln[\delta(i)] - \ln[|z|] + \frac{1}{\mathcal{S}(2\pi)^d}\int_B \ln\left[\mathcal{D}^{\mathcal{W}}(z,\theta)\right] d\theta$$

where

$$\mathcal{D}^{\mathcal{W}}(z,\theta) = \det[1 - z\,\Lambda(\theta)]$$

(ii) $T_w(z)$ satisfies to

$$-z\frac{dT_w}{dz}(z) = P(0,z) \tag{4.6}$$

with $T_w(z \to 1^-) = z_L$;
(iii) $T_w(z \to -1^-)$ is finite.
(iv) If for all $(a,b) \in E(L)$ $w(a,b) = 1$ and L is vertex transitive then

$$T_w(z) = T(z), \forall z \in (0,1]$$

where $T(z)$ is the STGF defined in [26].

Proof. The proof of i),ii) and iii) is just an adaptation of Theorem 2.4.2 . The proof of (iii) and iv) are trivial.
remark. Note that if $w = c$, and L is regular, then $T_w(z) = \ln[a] + T_e(z)$. And if L is vertex transitive then $\delta(i) = \delta(j), \forall i, j$.

4.5 DIMER FREE ENERGY, ANISOTROPIC ISING FREE ENERGY AND THE $wSTGF$ FOR THE SQUARE AND TRIANGLE LATTICE

In Theorem 4.5.2 and Theorem 4.5.3 that we will present in the next section, we will relate the anisotropic Ising free energy, the Dimer free energy and the $wSTGF$ for the square and the triangle. We present bellow a list of the 3 functions for each lattice that we need for those theorems.

- Square lattice.

  - Dimer free energy. The Dimer model for the Square Lattice is given by [116].

$$F_{DM}(z_1, z_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \ln 2[z_1^2 + z_2^2 + z_1^2 \cos[\theta_1] + z_2^2 \cos[\theta_2]]\, d\theta$$

  - Anisotropic Ising free energy. [28],[32].

$$f(K_1, K_2) = \ln[2] + \frac{1}{2(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathcal{A}(\theta_1, \theta_2)\, d\theta$$

    where

$$\mathcal{A}(\theta_1, \theta_2) = \ln[\cosh 2K_1 \cosh 2K_2 + \sinh[2K_i] \cos[\theta_1] + \sinh[2K_i] \cos[\theta_2]].$$

  - $wSTGF$.
    The $wSTGF$ for the Square lattice $T(z_1, z_2, z) = T_w(z)$ is given by (See section 4.6.)

$$T_w(z) = \ln[2a + 2b] - ln[|z|] + \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \ln[1 - \frac{z}{a+b}(a\cos[\theta_1] + b\cos[\theta_2]]\, d\theta$$

- Triangle lattice.

  – Dimer free energy. The Dimer free energy for the Triangle Lattice is given by [116].

$$F_{DM}(z_1,z_2,z_3)=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\ln 2\Big[\sum_{i=1}^{3}z_i^2+z_1^2\cos[\theta_1]+z_2^2\cos[\theta_2]+z_3^2\cos[\theta_1+\theta_2]\Big]\,d\theta$$

  – Anisotropic Ising free energy[32].

$$f(K_1,K_2,K_3)=\ln[2]+\frac{1}{2(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\mathcal{B}(\theta_1,\theta_2)\,d\theta,$$

  where

$$\begin{aligned}\mathcal{B}(\theta_1,\theta_2)&=\ln[C_1C_2C_3+S_1S_2S_3-S_1\cos[\theta_1]-S_2\cos[\theta_2]-S_3\cos[\theta_1+\theta_2]],\\ S_i&=\sinh[2K_i],\quad \forall i=1,2,3,\\ C_i&=\cosh[2K_i],\quad \forall i=1,2,3.\end{aligned}$$

  – $wSTGF$.

  The $wSTGF$ for the Triangle lattice $T(z_1,z_2,z_3,z)=T_w(z)$ is given by (See section 4.6.)

$$\begin{aligned}T_w(z)=&-ln[|z|]+\\ &\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln 2[a+b+c-z(a\cos[\theta_1]+b\cos[\theta_2]+c\cos[\theta_1+\theta_2]]\,d\theta\end{aligned}$$

- Hexagonal lattice

  – Anisotropic Ising free energy [32].

$$f(K_1,K_2,K_3)=\frac{3}{4}\ln[2]+\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathcal{C}(\theta_1,\theta_2)\,d\theta$$

  where

$$\begin{aligned}\mathcal{C}(\theta_1,\theta_2)&=\ln[C_1C_2C_3+1-S_2S_3\cos[\theta_1]-S_3S_1\cos[\theta_2]-S_1S_2\cos[\theta_1-\theta_2]],\\ S_i&=\sinh[2K_i],\quad \forall i=1,2,3,\\ C_i&=\cosh[2K_i],\quad \forall i=1,2,3.\end{aligned}$$

  – $wSTGF$

$$T_w(z)=\ln[a+b+c]-ln[|z|]+\frac{1}{2(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\ln[\mathcal{D}^{\mathcal{W}}(z,\theta)]\,d\theta$$

  where

$$\mathcal{D}^{\mathcal{W}}(z,\theta)=1-\frac{2z^2}{a^2+b^2+c^2}(ac\cos[\theta_1]+bc\cos[\theta_2]+ab\cos[\theta_1+\theta_2]$$

4.5.1 Dimer model and the $wSTGF$.

Theorem 4.5.1 *Let $T(z_1, \dots, z_n, z)$, $F_{DM}(z_1, \dots, z_n)$ the $wSTGF$, and Dimer free energy for the square lattice ($n = 2$) or the triangle lattice ($n = 3$) then*

$$T(z_1^2, ..., z_n^2, -1) = 2F_{DM}(z_1, ..., z_n) + \ln[2]$$

Proof. The proof follows just by looking at the expressions for the Dimer free energy and the $wSTFG$ given in section 4.5. .

4.5.2 Anisotropic Ising model (Connections.)

Theorem 4.5.2 *Let $T(a_1, \dots, a_n, Z)$ and $f(K_1, \dots, K_n)$ the $wSTGF$, and the anisotropic Ising free energy for the Square lattice ($n = 2$) or the triangle lattice ($n = 3$), then*

$$f(K_1, \dots, K_n) = \frac{1}{2}T(a_1, \dots, a_n, Z) + \frac{1}{2}\ln[2Z],$$

where

$$\begin{aligned} a_i[K_i] =& \frac{\sinh[2K_i]}{Z}, \qquad\qquad \forall i = 1, \dots, n \\ Z[K_1, \dots, K_n] =& \frac{\sum_{i=1}^{n} \sinh[2K_i]}{\prod_{i=1}^{n} \cosh[2K_i] + (n-2)\prod_{i=1}^{n} \sinh[2K_i]}. \end{aligned}$$

Proof. The proof follows just by looking at the expressions for the Dimer free energy and the $wSTFG$ given in section 4.5. .

Theorem 4.5.3 *Let $T(a_1, a_3, a_3, Z)$ and $f(K_1, K_2, K_3)$ the $wSTGF$, and the anisotropic Ising free energy for the Hexagonal lattice, then*

$$f(K_1, K_2, K_3) = \frac{1}{2}T(a_1, a_2, a_3, Z) + \frac{1}{2}\ln[Z] + \frac{3}{4}\ln[2],$$

where

$$\begin{aligned} a_i[K_i] =& \frac{\sinh[2K_i]}{Z}, \qquad\qquad \forall i = 1, 2, 3 \\ Z[K_1, K_2, K_3] =& \sqrt{\frac{(\sum_{i=1}^{3} \sinh[2K_i])^2}{2(\prod_{i=1}^{3} \cosh[2K_i] + 1) + \sum_{i=1}^{3} \sinh^2[2K_i]}}. \end{aligned}$$

Proof. The proof follows just by looking at the expressions for the Dimer free energy and the $wSTFG$ given in section 4.5. .

We remark that if $K's$ are equal in the above theorem we obtain the results for the isotropic Ising model, obtained for the case of the square in [27] and for triangle and hexagonal lattices in chapter 3 (Theorem 3.6.2),

4.6 $wSTGF$ FOR THE SQUARE, TRIANGLE, UNION JACK AND CAIRO PENTAGONAL LATTICE

In this section we present the $wSTGF$ for the square triangle, union jack and cairo pentagonal lattices, according to the Theorem 4.4.1. For the case of the square and traingle lattices we present the full details to obtain the final expression of the $wSTGF$.

4.6.1 $T_w(z)$ for the Square lattice

Given the square lattice we can define a periodic weight w with parameters $a, b \in (0, \infty)$ as in fig. 32. In this case

$$w[(0,0),(1,0)] = w[(0,0),(-1,0)] = a$$
$$w[(0,0),(0,1)] = w[(0,0),(0,-1)] = b$$

$$\delta(0,0) = \sum_{i \in \{1,-1\}} w[(0,0),(i,0)] + w[(0,0),(0,i)]$$

so $\delta(0,0) = 2a + 2b$,

$$p(1,0) = p(-1,0) = \frac{a}{2(a+b)}$$
$$p(0,1) = p(0,1) = \frac{b}{2(a+b)}$$

$$\Lambda[k] = p(1,0)e^{i\theta_1} + p(-1,0)e^{-i\theta_1} + p(1,0)e^{i\theta_2} + p(-1,0)e^{-i\theta_2}$$

so

$$\Lambda[k] = \frac{1}{a+b}(a\cos[\theta_1] + b\cos[\theta_2]),$$

finally by Theorem 2

$$T_w(z) = \ln[2a+2b] - ln[|z|] + \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \ln[1 - \frac{z}{a+b}(a\cos[\theta_1] + b\cos[\theta_2]]\,d\theta \quad (4.7)$$

We can write Eq. (4.7) as follows

$$T_w(z) = -ln[|z|] + \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln 2[a + b - z(a\cos[\theta_1] + b\cos[\theta_2])]\,d\theta \quad (4.8)$$

$$f(K_1, K_2) = \ln[2] + \frac{1}{2(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \ln[\cosh[2K_1]\cosh[2K_2] + \sin[2K_1]\cos[\theta_1] + \sin[2K_2]\cos[\theta_2]]\,d\theta$$

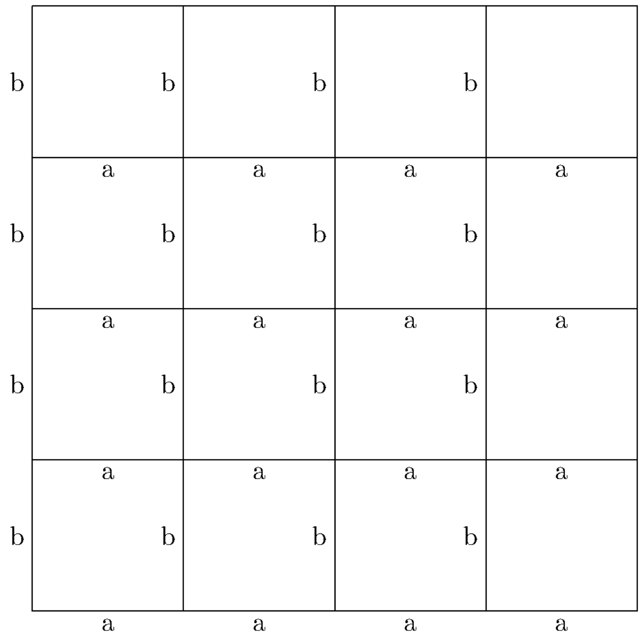


Figure 32 – Square lattice with a periodic weight given by 2 real numbers positive numbers a,b.

4.6.2 $T_w(z)$ for the triangle lattice

Given the Triangle lattice we can define a periodic weight w with parameters $a, b, c \in (0, \infty)$ as in fig. 33. In this case

$$w[(0,0),(1,0)] = w[(0,0),(-1,0)] = a$$

$$w[(0,0),(0,1)] = w[(0,0),(0,-1)] = b$$

$$w[(0,0),(1,1)] = w[(0,0),(-1,-1)] = c$$

$$\delta(0,0) = \sum_{i\in\{1,-1\}} w[(0,0),(i,0)] + w[(0,0),(0,i)] + w[(0,0),(i,i)]$$

so $\delta(0,0) = 2a + 2b + 2c$,

$$p(1,0) = p(-1,0) = \frac{a}{2(a+b+c)}$$

$$p(0,1) = p(0,1) = \frac{b}{2(a+b+c)}$$

$$p(1,1) = p(-1,-1) = \frac{c}{2(a+b+c)}$$

$$\lambda[k] = p(1,0)e^{i\theta_1} + p(-1,0)e^{-i\theta_1} + p(1,0)e^{i\theta_2} + p(-1,0)e^{-i\theta_2} + p(1,1)e^{i(\theta_1+\theta_2)} + p(-1,-1)e^{-i(\theta_1+\theta_2)}$$

so

$$\Lambda[k] = \frac{1}{a+b+c}(a\cos[\theta_1] + b\cos[\theta_2] + c\cos[\theta_1+\theta_2]],$$

finally by Theorem 2

$$T_w(z) = \ln[2a+2b+2c] - ln[|z|] + \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \ln[1 - \frac{z}{a+b+c}(a\cos[\theta_1] + b\cos[\theta_2] + c\cos[\theta_1+\theta_2]]\, d\theta \quad (4.9)$$

We can write Eq. (4.7) as follows

$$T_w(z) = -ln[|z|] + \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln 2[a+b+c - z(a\cos[\theta_1] + b\cos[\theta_2] + c\cos[\theta_1+\theta_2]]\, d\theta \quad (4.10)$$

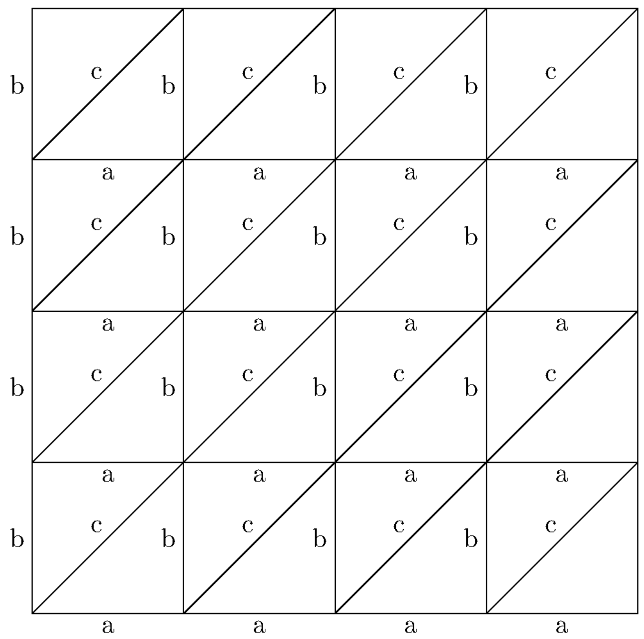


Figure 33 – Triangle lattice with a periodic weight given by 3 real numbers non-negative numbers a,b,c.

4.6.3 $T_w(z)$ for the Union Jack Lattice.

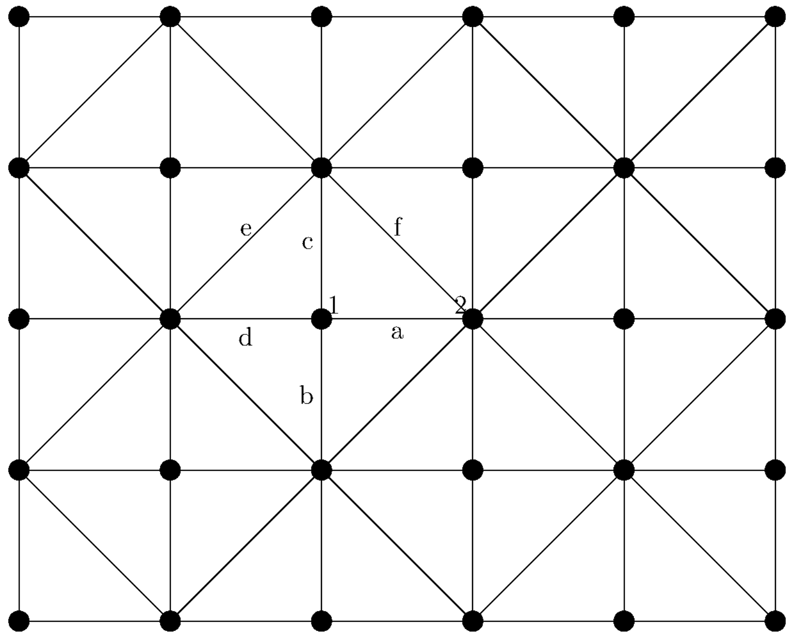


Figure 34 – Union Jack Lattice with periodic weights.$\mathcal{S} = 2, \mathcal{E} = 6.$

$$D^{\mathcal{W}}_{Un}(x, y) = a_0 + a_1 \cos(x) + a_2 \cos(y) + a_3 \cos(x + y) + a_4 \cos(x - y)$$

$$\begin{aligned}
a_0 &= -a^2z^2 - b^2z^2 - c^2z^2 - d^2z^2 + q_1q_2,\\
a_1 &= -2abz^2 - 2cdz^2 - 2eq_1z,\\
a_2 &= -2acz^2 - 2bdz^2 - 2fq_1z,\\
a_3 &= -2bcz^2,\\
a_4 &= -2adz^2.
\end{aligned}$$

where $q_1 = a + b + c + d$, and $q_2 = e + f$.

4.6.4 $T_w(z)$ for the Cairo lattice

$$D^{\mathcal{W}}_{cr}(x, y) = a_0 + a_1 \cos(x) + a_2 \cos(y) + a_3 \cos(2x) + a_4 \cos(2y) + a_5 \cos(x - y) + a_6 \cos(x + y)$$

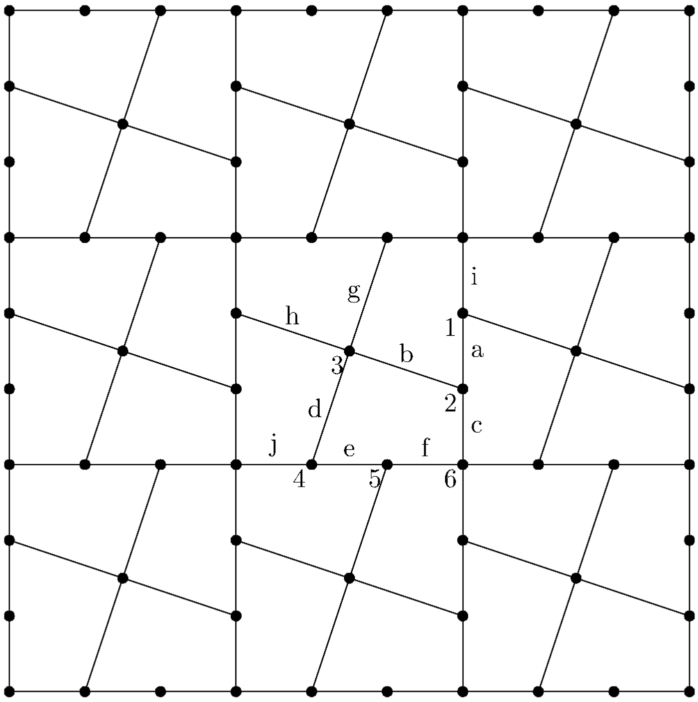


Figure 35 – Cairo Lattice with periodic weights. $\mathcal{S} = 5, \mathcal{E} = 8$.

$$
\begin{aligned}
a_0 =& - a^2d^2f^2z^6 + a^2d^2q_5q_6z^4 + a^2e^2q_3q_6z^4 + a^2f^2q_3q_4z^4 - a^2g^2j^2z^6 + a^2g^2q_4q_6z^4 \\
& + a^2j^2q_3q_5z^4 - a^2q_3q_4q_5q_6z^2 + 2abefhjz^6 - 2abfgiq_4z^5 + 2acdegiz^6 - 2acdhjq_5z^5 \\
& - b^2e^2i^2z^6 + b^2e^2q_1q_6z^4 + b^2f^2q_1q_4z^4 + b^2i^2q_4q_5z^4 + b^2j^2q_1q_5z^4 - b^2q_1q_4q_5q_6z^2 \\
& - 2bcdefq_1z^5 + c^2d^2q_1q_5z^4 - c^2e^2h^2z^6 + c^2e^2q_1q_3z^4 + c^2g^2q_1q_4z^4 + c^2h^2q_4q_5z^4 \\
& - c^2q_1q_3q_4q_5z^2 + d^2f^2q_1q_2z^4 + d^2i^2q_2q_5z^4 - d^2q_1q_2q_5q_6z^2 + e^2h^2q_2q_6z^4 + e^2i^2q_2q_3z^4 \\
& - e^2q_1q_2q_3q_6z^2 - 2eghijq_2z^5 + f^2h^2q_2q_4z^4 - f^2q_1q_2q_3q_4z^2 + g^2i^2q_2q_4z^4 + g^2j^2q_1q_2z^4 \\
& - g^2q_1q_2q_4q_6z^2 + h^2j^2q_2q_5z^4 - h^2q_2q_4q_5q_6z^2 - i^2q_2q_3q_4q_5z^2 - j^2q_1q_2q_3q_5z^2 + q_1q_2q_3q_4q_5q_6, \\
a_1 =& 2a^2efjq_3z^5 + 2abe^2hq_6z^5 - 2abegijz^6 + 2abf^2hq_4z^5 + 2abhj^2q_5z^5 - 2abhq_4q_5q_6z^3 \\
& - 2acdefhz^6 + 2b^2efjq_1z^5 - 2bcdjq_1q_5z^4 + 2efh^2jq_2z^5 - 2efjq_1q_2q_3z^3 - 2fghiq_2q_4z^4, \\
a_2 =& 2a^2degq_6z^5 - 2abdefiz^6 + 2acd^2iq_5z^5 + 2ace^2iq_3z^5 - 2aceghjz^6 + 2acg^2iq_4z^5 - 2aciq_3q_4q_5z^3 \\
& - 2bcfgq_1q_4z^4 + 2c^2degq_1z^5 + 2degi^2q_2z^5 - 2degq_1q_2q_6z^3 - 2dhijq_2q_5z^4, \\
a_3 =& 2abefhjz^6, \\
a_4 =& 2acdegiz^6, \\
a_5 =& 2bce^2hiz^6 - 2bcegjq_1z^5 - 2bchiq_4q_5z^4 - 2defhiq_2z^5, \\
a_6 =& 2a^2dfgjz^6 - 2abdijq_5z^5 - 2acfghq_4z^5 - 2dfgjq_1q_2z^4.
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= i + a + h, \\
q_2 &= a + c + b, \\
q_3 &= d + g + b + h, \\
q_4 &= d + j + e, \\
q_5 &= e + f + g, \\
q_6 &= c + f + i + j.
\end{aligned}
$$

4.7 CONCLUSION

In this chapter we have extended the formula (and notion) for the $eSTGF$ given in Chapter 2 (Theorem 2.4.2) to a more general setting where we allow non-regular periodic lattices with positive weights on every edge. We call this new function as $wSTGF$ (w for weighted), the formula for the $wSTGF$ is given in Theorem 2, furthermore we relate this function to the Dimer free energy(Theorem 4.5.1 on the square and triangle lattices and to the anisotropic Ising model on the square, triangle and hexagonal lattices(Theorems 4.5.2 and 4.5.3). Note that in chapter 3 we generalized the relation between the Ising model and spanning trees reported in [27] to other lattices, but in this final chapter we follow a different route because our generalization was in the direction of allowing general couplings on the Ising model (Theorem 4.5.2 and Theorem). We also calculated explicitly the $wSTGF$ for the square, triangle and for to important non-regular lattices: the Union jack lattice and the Cairo pentagonal lattices.

5 CONCLUSIONS

A. Main results achieved

In this Thesis we have generally addressed the problem of solving lattice models (defined on a planar graph G) in Statistical Physics using the combinatorial idea of counting the number of spanning trees on G. With this aim, we have revised many aspects of rigorous graph theory as well as discussed distinct topological features of random walks on periodic graphs. In particular, we have proposed new extensions of the Spanning Tree Generating Function (STGF) originally developed in [26]. Considering such functions, we have derived mappings associating the corresponding STGF to the partition function and free energy of some representative models commonly studied in the literature.

Our main findings have been divided into three major Chapters, each describing a specific group of results. They have been organized as explained below. But before listing each one of them, we can summarize the essential core of the present work as the following.

We have obtained relations between the isotropic Ising free energy and the spanning tree generating function (STGF) for all the Archimedean lattices (an important set of vertex-transitive periodic planar lattices). Also, for an important non-vertex transitive lattice, called the Martini lattice, we have done the same but then employing a new extended spanning tree generating function $(eSTGF)$ proposed in this Thesis. We have further found a relation between a weighted Spanning Tree Generating Function $(wSTGF)$ and two lattice models on the square and triangle lattices, namely the anisotropic Ising model and the Dimer model. This was inspired by the conjecture in Ref. [27].

In Chapter 2 we have developed in Theorem 2.4.2 a novel integral formula for the $STGF$, which was defined only for vertex-transitive periodic lattices in [26] via a differential equation. Furthermore, in this theorem we have extend the definition of $STGF$ (which we call the $eSTGF$), thus including regular non-vertex transitive lattices. We have provided explicit calculations of our $eSTGF$ expression for the eleven Archimedean lattices, which are examples of vertex-transitive graphs (we emphasize that in this case $STGF = eSTGF$). We should remark that in 2D, the expressions for the $STGF$ previously reported in the literature where only for the square, triangle and hexagonal lattices [26]. In this way, most of the expressions for the $STGF$ presented in this thesis are completely new. We also have treated non-vertex transitive lattices, obtaining the

$eSTGF$ for the Martini and medial lattices. Lastly, regarding the $eSTGF$, thanks to theorem 2.3.1 in Corollary 2.4.2.1 we have obtained an original handy relation for the Lattice Green fucntion $P(z, 0)$ of a vertex-transitive lattice.

In Chapter 2 we also have thoroughly introduced the RWLS model on regular periodic lattices G, moreover generalizing some formal results associated to it in the literature. We have proved a new result (theorem 2.6.2), which essentially shows that the $eSTGF$ of G yields the free-energy of a loop soup model on the same G.

In Chapter 3 we have first calculated the formulas for the isotropic Ising free energy for all the eleven Archimedean lattices. Our results agree with those isotropic Ising free energy already published in the literature, namely, the square, triangle, hexagonal, kagome, bathroom and star lattices [28, 32, 33]. To the best of our knowledge, the isotropic Ising free energy derived here for the other 5 Archimedean lattices have not been previously reported anywhere. From the explicit expressions for the $STGF$ computed for the Archimedean lattices in Chapter 2, we have demonstrated important relations between each $STGF$ and the related isotropic Ising free energy.

Theorem 3.6.2 and Corollary 3.6.4 are our main results in this Chapter. They state that we can reconstruct the isotropic Ising free energy for any Archimedean case using its $STGF$ via a set of auxiliary real analytic functions. This new contribution extends the relation between the $STGF$ and the Ising free energy of the square lattice given by Theorem 1 in [27]. In fact, we propose a new conjecture, that $STGF$ and isotropic Ising free energy can be mapped into each other for any vertex-transitive periodic lattice.

Finally, at the end of Chapter 3 we have proved a result similar to Theorem 1, but for the particular (and important) case of the non-vertex transitive Martini lattice. Such theorem 3.7.1 is a step forward in trying to generalize the hypothesis in [27] to a more general class of periodic graphs.

Finally, in Chapter 4 we have extended the notion of $eSTGF$, given in Chapter 2 (Theorem 2.4.2), to an even more universal setting, where we allow non-regular periodic lattices with positive weights on every edge. This is probably the most general situation possible for a periodic lattice. We have called this new function the $wSTGF$ (w for weighted). The formula for our $wSTGF$ is given as Theorem 4.4.1. As illustrations we have considered the $wSTGF$ for the square, triangle and two important non-regular examples: the union jack and the Cairo pentagonal lattices.

For the cases of square and triangular lattices, we have related the $wSTGF$ to the Dimer model free energy (Theorem 4.5.1). This has also been done for the

Anisotropic Ising model for the case of the square, triangle and hexagonal lattices (Theorem 4.5.2 and Theorem 4.5.3).

We should observe that in Chapter 3 we have generalized the relation between the Ising model and spanning trees, previously determined only for the square and triangular lattices [27], to all the other Archimedean lattices by means of direct comparison between the expressions and then deriving appropriate mappings. Nonetheless, in this Chapter 4 we have followed a different route, towards allowing general couplings for the Ising model, hence we worked with the anisotropic Ising model. (Theorem 4.5.2 and Theorem 4.5.3).

B. Potential continuations for the present study

Many interesting questions have emerged while developing the present Thesis and unfortunately we have no time to analyze them in more details. Therefore, we believe some findings here could be the starting point for future studies addressing different aspects of the relevant Ising and Dimer models.

In this way, further work could be focused on the following issues:

- Try to define a notion of $STGF$ on lattices whose quotient can be embedded on a bi-torus or sphere, or any other surface with a different topology.
- By Corollary 2 in Chapter 3 for a grid Z^d, with $d = 1, 2$, we have the equation
$$f_L(K) = \ln \sqrt{2 \sinh[2K]} + \frac{1}{d} T(\phi_L(K)),$$
where $f_L(K)$ is the isotropic Ising free-energy, $T(z)$ is the STGF and $\phi_L(K)$ is an auxiliary function, mapping the two quantities. It should be really a great result (related to the Ising model in 3D) to extend this equation for the case of $d > 2$. In other words, to derive a proper mapping between isotropic Ising free-energy and $STGF$ for any hypercube.
- To extend the formula in Corollary 3 of Chapter 3 to any vertex-transitive periodic lattice, which presently are applied only to the eleven Archimedean lattices.
- To find a formula that relates the $wSTGF$ for a lattice and its dual.
- Although not clear to us about potential physical interpretations, mathematically it would be very interesting trying to generalize the expression for the $wSTGF$ given in Theorem 4.4.1 to settings like complex weights in each edge and for bigger domains in z (maybe $z \in C$). Also, try to extend $wSTGF$ for directed lattices and eventually to non-periodic lattices such as the Bethe [19] and Penrose [18] lattices.

Perhaps, these more universal constructions would allow the analytical solution of other more involving lattice models.

- From Theorem 4.5.1 we have obtained the following relation between the Dimer model free-energy F_{DM} and the $wSTGF$ for the square $(n = 2)$ and triangular $(n = 3)$ lattices

$$F_{DM}(z_1, ..., z_n) = \frac{1}{2}\hat{T}(z_1^2, ..., z_n^2, -1) + \frac{1}{2}\ln[2].$$

Of course, a natural step would be to look for a similar relation for other lattices.

- Finally, Theorem 4.5.2 leads to the following relation between the anisotropic Ising model of free-energy $f(K_1, ..., K_n)$, and the $wSTGF$ for the square $(n = 2)$ and triangular $(n = 3)$ lattices

$$f(K_1, ..., K_n) = \frac{1}{2}\hat{T}(a_1, .., a_n, z) + \frac{1}{2}\ln[2z],$$

where a_i $(i = 1, \ldots, n)$ and z are auxiliary functions of the variables $K_1, \ldots, K_n$. Futhermore, we obtained a similar relation for the hexagonal lattice in Theorem 4.5.3. So an important question is how to obtain akin formulas for other lattices.

REFERENCES

[1] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*. New York: W. H. Freeman & Co., 1986.

[2] D. Z. Zhang, "C.n. yang and contemporary mathematics," in *Mathematical Conversations*, pp. 61–73, Springer New York, 2001.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*. New York: Elsevier, 1976.

[4] B. Bollobás, *Modern Graph Theory*. Heidelberg: Springer, 1998.

[5] T. Sunada, *Topological Crystallography: With a View Towards Discrete Geometric Analysis*. Surveys and Tutorials in the Applied Mathematical Sciences, Tokyo: Springer, 2012.

[6] W. Woess, *Random Walks on Infinite Graphs and Groups*. Cambridge: Cambridge University Press, 2000.

[7] L. Lovász, L. Lovasz, A. Gyárfás, G. Katona, and A. Recski, *Graph Theory and Combinatorial Biology*. Bolyai Society mathematical studies, János Bolyai Mathematical Society, 1999.

[8] W. Kocay and D. L. Kreher, *Graphs, Algorithms and Optimization*. Chapman & Hall/CRC, 2004.

[9] M. Aigner, *A Course in Enumeration*. Berlin: Springer, 2007.

[10] A. Barabási and M. Pósfai, *Network science*. Cambridge: Cambridge University Press, 2016.

[11] M. A. Porter and J. P. Gleeson, *Dynamical systems on networks: a tutorial*. Frontiers in applied dynamical systems reviews and tutorials, Cham: Springer, 2016.

[12] F. Harary, F. NATO Advanced Study Institute (1963 : Paris, and . NATO Advanced Study Institute, Paris, *Graph Theory and Theoretical Physics*. Academic P., 1967.

[13] J. W. Essam, "Graph theory and statistical physics," *Discrete Mathematics*, vol. 1, pp. 83–112, 1971.

[14] L. Beaudin, J. Ellis-Monaghan, G. Pangborn, and R. Shrock, "A little statistical mechanics for the graph theorist," *Discrete Math.*, vol. 310, pp. 2037–2053, 2010.

[15] J. A. Ellis-Monaghan and C. Merino, "Graph polynomials and their applications i: The tutte polynomial," in *Structural Analysis of Complex Networks*, pp. 219–255, Birkhäuser Boston, 2010.

[16] R. Twarock and A. Luque, "Structural puzzles in virology solved with an overarching icosahedral design principle," *Nat. Commun.*, vol. 10, p. 4414, 2019.

[17] R. Gajić, . Jovanović, K. Hingerl, R. Meisels, and F. Kuchar, "2d photonic crystals on the archimedean lattices (tribute to johannes kepler (1571–1630))," *Optical Materials*, vol. 30, pp. 1065–1069, 2008.

[18] E. S. Sørensen, M. V. Jarić, and M. Ronchetti, "Ising model on penrose lattices: Boundary conditions," *Phys. Rev. B*, vol. 44, pp. 9271–9282, 1991.

[19] M. Ostilli, "Cayley trees and bethe lattices: A concise analysis for mathematicians and physicists," *Physica A: Statistical Mechanics and its Applications*, vol. 391, pp. 3417–3423, 2012.

[20] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, 1847.

[21] C. Fortuin and P. Kasteleyn, "On the random-cluster model: I. introduction and relation to other models," *Physica*, vol. 57, pp. 536–564, 1972.

[22] F. Y. Wu, "The potts model," *Rev. Mod. Phys.*, vol. 54, pp. 235–268, 1982.

[23] R. Lyons, "Asymptotic enumeration of spanning trees," *Combin. Probab. Comput.*, vol. 14, pp. 491–522, 2005.

[24] S.-C. Chang and W. Wang, "Spanning trees on lattices and integral identities," *J. Phys. A: Math. Gen.*, vol. 39, pp. 10263–10275, 2006.

[25] R. Shrock and F. Y. Wu, "Spanning trees on graphs and lattices in d dimensions," *J. Phys. A*, vol. 33, pp. 3881–3902, 2000.

[26] A. Guttmann and M. Rogers, "Spanning tree generating functions and mahler measures," *J. Phys. A*, vol. 45, p. 494001, 2012.

[27] G. M. Viswanathan, "Correspondence between spanning trees and the Ising model on a square lattice," *Phys. Rev. E*, vol. 95, p. 062138, 2017.

[28] L. Onsager, "Crystal statistics. i. a two-dimensional model with an order-disorder transition," *Phys. Rev.*, vol. 65, pp. 117–149, 1944.

[29] J. C. W. M. Kac, “A combinatorial solution of the two-dimensional ising model,” *Phys. Rev.*, vol. 88, p. 1332, 1952.

[30] R. Feynman, *Statistical Mechanics: A Set Of Lectures*. Boca Raton: CRC Press, 2018.

[31] S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge: Cambridge University Press, 2017.

[32] C. Domb, “On the theory of cooperative phenomena in crystals,” *Advances in Physics*, vol. 9, pp. 149–244, 1960.

[33] V. Matveev and R. Shrock, “Complex-temperature properties of the ising model on 2d heteropolygonal lattices,” *Journal of Physics A: Mathematical and General*, vol. 28, pp. 5235–5256, 1995.

[34] P. W. Kasteleyn, “Dimer statistics and phase transitions,” *Journal of Mathematical Physics*, vol. 4, pp. 287–293, 1963.

[35] P. W. Kasteleyn, “Graph theory and cristal physics,” in *Graph Theory and Theoretical Physics*, pp. 43–110, London: Academic Press, 1967.

[36] D. Cimasoni and N. Reshetikhin, “Dimers on surface graphs and spin structures. i,” *Communications in Mathematical Physics*, vol. 275, pp. 187–208, 2007.

[37] F. Camia, “Scaling limits, brownian loops, and conformal fields,” in *Advances in Disordered Systems, Random Processes and Some Applications* (P. Contucci and C. Giardinà, eds.), pp. 205–269, Cambridge: Cambridge University Press, 2016.

[38] G. F. Lawler and J. A. T. Ferreras, “Random walk loop soup,” *Trans. Amer. Math. Soc.*, vol. 359, pp. 767–787, 2007.

[39] Y. Le Jan, “Markov loops and renormalization,” *Ann. Probab.*, vol. 38, pp. 1280–1319, 2010.

[40] T. van de Brug, F. Camia, and M. Lis, “Random walk loop soups and conformal loop ensembles,” *Probab. Theory Relat. Fields*, vol. 166, pp. 553–584, 2016.

[41] F. Camia and M. Lis, “Non-backtracking loop soups and statistical mechanics on spin networks,” *Ann. Henri Poincaré*, vol. 18, pp. 403–433, 2017.

[42] H. Ooguri, “Topological lattice models in four dimensions,” *Mod. Phys. Lett. A*, vol. 7, pp. 2799–2810, 1992.

[43] C. Hoelbling and C. B. Lang, “Universality of the ising model on spherelike lattices,” *Phys. Rev. B*, vol. 54, pp. 3434–3441, 1996.

[44] M. Bauer and D. Bernard, “2d growth processes: SLE and Loewner chains,” *Phys. Rep.*, vol. 432, pp. 115–221, 2006.

[45] E. Estrada, “Graph and network theory in physics,” 2013.

[46] G. Grimmett, *Probability on Graphs: Random Processes on Graphs and Lattices*. Institute of Mathematical Statistics Textbooks, Cambridge: Cambridge University Press, 2 ed., 2018.

[47] G. Mo-lin, P. H, and W. Yueh, *Lattice Statistics And Mathematical Physics: Festschrift Dedicated To Professor Fa-yueh Wu On The Occasion Of His 70th Birthday, Proceedings Of Apctp-nankai Joint Symposium*. Series On Advances In Statistical Mechanics, Singapore: World Scientific Publishing Company, 2002.

[48] T. Degushi, “Introduction to solvable lattice models in statistical and mathematical physics,” in *Classical and Quantum Integrable Systems: Theory and Applications* (A. Kundu, ed.), ch. 5, pp. 113–151, Bristol: IOP Publishing, 2003.

[49] M. Kardar, *Statistical Physics of Particles*. Cambridge: Cambridge University Press, 2007.

[50] S. Katsura, T. Morita, S. Inawashiro, T. Horiguchi, and Y. Abe, “Lattice Green’s function. Introduction,” *J. Math Phys.*, vol. 12, pp. 892–895, 1971.

[51] E. D. Ocansey, “Enumeration problems on lattices,” m.S. thesis, Stellenbosch University, 2013.

[52] J.-M. Choi, A. I. Gilson, and E. I. Shakhnovich, “Graph’s topology and free energy of a spin model on the graph,” *Phys. Rev. Lett.*, vol. 118, p. 088302, 2017.

[53] B. Hughes and S. Hughes, *Random Walks and Random Environments: Random walks*. No. v. 1 in Oxford science publications, Clarendon Press, 1995.

[54] A. J. Guttmann, “Lattice Green’s functions in all dimensions,” *J. Phys. A*, vol. 43, p. 305205, 2010.

[55] W. Kager, M. Lis, and R. W. J. Meester, “The signed loop approach to the ising model: foundations and critical point,” *J. Stat. Phys.*, vol. 152, pp. 353–387, 2013.

[56] B. de Tilière, “From cycle rooted spanning forests to the critical Ising model: an explicit construction,” *Comm. Math. Phys.*, vol. 319, pp. 69–110, 2013.

[57] F. Y. Wu, "Number of spanning trees on a lattice," *J. Phys. A*, vol. 10, pp. L113–L115, 1977.

[58] B. R. Myers, "Generating function for spanning trees," *Electron. Lett.*, vol. 9, pp. 360–361, 1973.

[59] A. Rosengren, "On the number of spanning trees for the 3D simple cubic lattice," *J. Phys. A*, vol. 20, pp. L923–L927, 1987.

[60] G. S. Joyce, R. T. Delves, and I. J. Zucker, "Exact evaluation of the Baxter-Bazhanov Green function," *J. Phys. A*, vol. 31, pp. 1781–1790, 1998.

[61] G. S. Joyce, "Singular behaviour of the cubic lattice Green functions and associated integrals," *J. Phys. A*, vol. 34, pp. 3831–3839, 2001.

[62] M. L. Glasser and G. Lamb, "A lattice spanning-tree entropy function," *J. Phys. A*, vol. 38, pp. L471–L475, 2005.

[63] A. D. Mednykh and I. A. Mednykh, "On Rationality of Generating Function for the Number of Spanning Trees in Circulant Graphs," *Algebra Colloq.*, vol. 27, pp. 87–94, 2020.

[64] H. S. Wilf, *Generatingfunctionology*. Wellesley: A.K. Peters, 2006.

[65] R. Burton and R. Pemantle, "Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances," *Ann. Probab.*, vol. 21, pp. 1329–1371, 1993.

[66] S.-C. Chang, "Spanning trees on the two-dimensional lattices with more than one type of vertex," *J. Phys. A: Math. Theor.*, vol. 42, p. 015208, 2008.

[67] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge: Cambridge University Press, 1985.

[68] S. Hassani, C. Koutschan, J.-M. Maillard, and N. Zenine, "Lattice Green functions: the d-dimensional face-centered cubic lattice, d= 8, 9, 10, 11, 12," *J. Phys. A*, vol. 49, p. 164003, 2016.

[69] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*. London: Springer, 1999.

[70] C. Smyth, "The Mahler measure of algebraic numbers: a survey," in *Number theory and polynomials*, vol. 352 of *London Math. Soc. Lecture Note Ser.*, pp. 322–349, Cambridge: Cambridge Univercity Press, 2008.

[71] M.-J. Bertin and M. Lalín, "Mahler measure of multivariable polynomials," in *Women in numbers 2: research directions in number theory*, vol. 606 of *Contemp. Math.*, pp. 125–147, Providence: Amer. Math. Soc., 2013.

[72] D. S. Silver and S. G. Williams, "Graph complexity and mahler measure," 2017.

[73] C. Scullard, "Exact site percolation thresholds using the site-to-bond and star-triangle transformations," *Phys. Rev. E*, vol. 73, p. 016107, 2006.

[74] C. Scullard and R. Ziff, "Predictions of bond percolation thresholds for the kagome and Archimedean (3,12(2)) lattices," *Phys. Rev. E*, vol. 73, p. 045102, 2006.

[75] C. Ding, Z. Fu, and W. Guo, "Critical points of the o(n) loop model on the martini and the 3-12 lattices," *Phys. Rev. E*, vol. 85, p. 062101, 2012.

[76] R. Parviainen, "Estimation of bond percolation thresholds on the Archimedean lattices," *J. Phys. A*, vol. 40, pp. 9253–9258, 2007.

[77] R. A. Neher, K. Mecke, and H. Wagner, "Topological estimation of percolation thresholds," *J. Stat. Mech.*, vol. 2008, p. P01011, 2008.

[78] R. Siddharthan and A. Georges, "Square kagome quantum antiferromagnet and the eight-vertex model," *Phys. Rev. B*, vol. 65, p. 014417, 2001.

[79] H. Nakano, Y. Hasegawa, and T. Sakai, "Magnetization jump in the magnetization process of the spin-1/2 heisenberg antiferromagnet on a distorted square-kagome lattice," *J. Phys. Soc. Jpn.*, vol. 84, p. 114703, 2015.

[80] J. L. Jacobsen, "High-precision percolation thresholds and potts-model critical manifolds from graph polynomials," *J. Phys. A*, vol. 47, p. 135001, 2014.

[81] J.-M. Couvreur, D. Poitrenaud, and P. Weil, "Branching processes of general petri nets," in *Applications and Theory of Petri Nets* (L. M. Kristensen and L. Petrucci, eds.), (Berlin), pp. 129–148, Springer, 2011.

[82] M. Kotani, T. Shirai, and T. Sunada, "Asymptotic behavior of the transition probability of a random walk on an infinite graph," *J. Funct. Anal.*, vol. 159, pp. 664–689, 1998.

[83] G. S. Joyce, "On the cubic modular transformation and the cubic lattice Green functions," *J. Phys. A*, vol. 31, pp. 5105–5115, 1998.

[84] G. N. Watson, "Three triple integrals," *Quart. J. Math. Oxford Ser.*, vol. 10, pp. 266–276, 1939.

[85] B. Halpering and A. Sevrin (Eds.), *Quantum Theory of Condensed Matter: Proceedings of the 24th Solvay Conference on Physics*. Singapore: World Scientific, 2010.

[86] P. Coleman, *Introduction to Many-Body Physics*. Cambridge: Cambridge University Press, 2015.

[87] M. E. Fisher, “Renormalization group theory: Its basis and formulation in statistical physics,” *Rev. Mod. Phys.*, vol. 70, p. 653, 1998.

[88] J. Y. Fu, “On the landau theory of phase transitions: a hierarchical dynamic model,” *J. Phys.: Condens. Matter.*, vol. 25, p. 075903, 2013.

[89] S. M. Bhattacharjee and A. Khare, “Fifty years of the exact solution of the two-dimensional ising model by onsager,” *Curr. Sc.*, vol. 69, pp. 816–821, 1995.

[90] D. A. Lavis and G. M. Bell, *Statistical Mechanics of Lattice Systems 1: Closed-Form and Exact Solutions*. Berlin: Springer-Verlag, 1999.

[91] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems*. Singapore: World Scientific, 2004.

[92] R. J. Baxter, *Exactly Solved Models in Statistical Physics*. Minneola: Dover, 2007.

[93] L. Samaj and Z. Bjnok, *Introduction to the Statistical Physics of Integrable Many-Body Systems*. Cambridge U: Cambridge University Press, 2013.

[94] D. P. Landau and K. Binder, *Monte Carlo Simulations in Statistical Physics, 4th ed.* Cambridge: Cambridge University Press, 2014.

[95] A. G. Percus, G. Istrate, and C. E. Moore, *Computational Complexity and Statistical Physics*. Oxford: Oxford University Press, 2006.

[96] C. d’Iribarne, G. Rasigni, and M. Rasigni, “Determination of site percolation transitions for 2d mosaics by means of minimal spanning tree approach,” *Phys. Lett. A*, vol. 209, pp. 95–98, 1995.

[97] R. Shrock and S. H. Tsai, “Ground-state entropy of potts antiferromagnets: Bounds, series, and monte carlo measurements,” *Phys. Rev. E*, vol. 56, p. 2733, 1997.

[98] P. N. Suding and R. M. Ziff, “Site percolation thresholds for archimedean lattices,” *Phys. Rev. E*, vol. 60, p. 275, 1999.

[99] A. A. David, T. Fujii, E. Matioli, R. Sharma, S. Nakamura, S. P. DenBaars, C. Weisbuchc, and H. Benisty, “Gan light-emitting diodes with archimedean lattice photonic crystals,” *Appl. Phys. Lett.*, vol. 88, p. 073510, 2006.

[100] C. R. Scullard and R. M. Ziff, "Critical surfaces for general bond percolation problems," *Phys. Rev. Lett.*, vol. 100, p. 185701, 2008.

[101] A. Eddi, A. Decelle, E. Fort, and Y. Counder, "Archimedean lattices in the bound states of wave interacting particles," *EPL*, vol. 87, p. 540002, 2009.

[102] J. A. Millan, D. Ortiz, G. van Anders, and S. C. Glotzer, "Self-assembly of archimedean tilings with enthalpically and entropically patchy polygons," *ACS Nano*, vol. 8, pp. 2918–2928, 2014.

[103] F. C. de Lima, G. J. Ferreira, and R. H. Miwa, "Topological flat band, dirac fermions and quantum spin hall phase in 2d archimedean lattices," *Phys. Chem. Chem. Phys.*, vol. 21, p. 22344, 2019.

[104] C. R. Scullard and J. L. Jacobsen, "Bond percolation thresholds on archimedean lattices from critical polynomial roots," *Phys. Rev. Res.*, vol. 2, p. 012050(R), 2020.

[105] A. A. Kulkarni, E. Hanson, R. Zhang, K. Thornton, and P. V. Braun, "Archimedean lattices emerge in template-directed eutectic solidification," *Nature*, vol. 577, p. 355, 2020.

[106] S. S. David, A. Chelnikov, and J. Lourioz, "Isotropic photonic structures: Archimedean-like tillings and quasi-crystals," *IEEE J. Quant. Electr.*, vol. 37, p. 1427, 2001.

[107] A. K. Daniel, R. Alexander, and A. Miyake, "Computational universality of symmetry-protected topologically ordered cluster phases on 2d archimedean lattices," *Quantum*, vol. 4, p. 228, 2020.

[108] D. Cimasoni and H. Duminil-Copin, "The critical temperature for the Ising model on planar doubly periodic graphs," *Electron. J. Probab.*, vol. 18, pp. no. 44, 18, 2013.

[109] B. Kafman, "Crystal statistics. ii. partition function evaluated by spinor analysis," *Phys. Rev.*, vol. 76, p. 1232, 1949.

[110] R. Feynman, *Statistical Mechanics: A Set Of Lectures*. Boca Raton: CRC Press, 2018.

[111] N. V. Vdovichenko, "A calculation of the partition function for a plane dipole lattice," *Soviet Physics JETP*, vol. 20, pp. 477–488, 1965.

[112] B. L. van der Waerden, "Die lange reichweite der regelmäßigen atomanordnung in mischkristallen," *Zeitschrift für Physik*, vol. 118, pp. 473–488, 1941.

[113] T. Morita, “Justification of vdovichenko’s method for the ising model on a two-dimensional lattice,” *J. Phys. A: Math. Gen.*, vol. 19, p. 1197, 1986.

[114] A. Codello, “Exact curie temperature for the ising model on archimedean and laves lattices,” *J. Phys. A: Math. Theor.*, vol. 43, p. 399801, 2010.

[115] Y. Hirose, A. Oguchi, and Y. Fukumoto, “Quantum dimer model containing rokhsar-kivelson point expressed by spin-$\frac{1}{2}$ heisenberg antiferromagnets,” *Phys. Rev. B*, vol. 101, p. 174440, 2020.

[116] F. Y. Wu, “Dimers on two-dimensional lattices,” *Internat. J. Modern Phys. B*, vol. 20, pp. 5357–5371, 2006.

[117] P. Kasteleyn, “The statistics of dimers on a lattice: I. the number of dimer arrangements on a quadratic lattice,” *Physica*, vol. 27, pp. 1209–1225, 1961.

[118] S. Li and W. Yan, “Spanning trees and dimer problem on the cairo pentagonal lattice,” *Applied Mathematics and Computation*, vol. 337, pp. 34–40, 2018.

[119] T. Fritz, “Corrigendum to “Velocity polytopes of periodic graphs and a no-go theorem for digital physics” [Discrete Mathematics 313 (2013) 1289–1301] [mr3061113],” *Discrete Math.*, vol. 313, p. 2380, 2013.

[120] A. Badanin, E. Korotyaev, and N. Saburova, “Laplacians on periodic discrete graphs,” 2013.

[121] C. S. Borcea and I. Streinu, “Periodic frameworks and flexibility,” *Proc. R. Soc. Lond. Ser. A*, vol. 466, pp. 2633–2649, 2010.

[122] O. Delgado-Friedrichs, “Barycentric drawings of periodic graphs,” in *Graph Drawing* (G. Liotta, ed.), (Berlin), pp. 178–189, Springer, 2004.

[123] S. Krantz, *A Primer of Real Analytic Funtions.* Basel: Birkhäuser Verlag, 1992.

[124] W. Rudin, *Principles of Mathematical Analysis. 3rd.* New York: McGraw-Hill, 1976.

APPENDIX A – PERIODIC GRAPHS AND PROOF OF THEOREM 2.3.1 AND THEOREM 2.6.1.

A.1 PERIODIC GRAPHS AS STRUCTURES IN THE Z^d

Let $G = (V, E)$ be an undirected, simple, connected and locally finite graph. Recalling the notation in section 2.3 — for which the elements of $E(G)$ are denoted by (v_x, v_y) with $v_x, v_y \in V(G)$ — if G is also an infinite d–periodic graph with the vertex set given by $Z^d \times S$ for $S = \{1, ..., \mathcal{S}\}$, then $\forall\, k, l \in Z^d$ and $s, t \in S$ it reads

$$((k, s), (l, t)) \Leftrightarrow ((k - l, s), (0, t)). \tag{A.1}$$

For A_G representing the adjacency matrix of G, the above condition can be expressed as

$$A_G((k, s), (l, t)) = A_G((k - l, s), (0, t)), \tag{A.2}$$

i.e., the properties in Eqs. (A.1) and (A.2) are equivalent.

The above is an abstract definition. However, in modeling distinct problems the tradition is to work with actual graph structures embedded in R^n ($n \geq d$). We call these concrete constructions *spatial lattices*. Hence, the natural question is if all the findings in the present contribution, which are based on a general description of periodic graphs, are directly extended to spatial lattices. The answer is yes, as we show below (we refer to [119, 120, 121, 122] for important results).

For every $u \in R^n$ a translation $\tau_u : R^n \to R^n$ is defined as

$$\tau_u(v) = v + u.$$

Then, typical spatial lattices in physical models can be defined as the following. A spatial lattice $L_\tau = (V_\tau, E_\tau)$ is an undirected, simple, connected and locally finite graph, with the vertex set $V_\tau \subset R^n$ given by

$$V_\tau = \bigcup_{i=1}^{n} (K + v_i), \qquad K = \Big\{ \sum_{i=1}^{d} k_i\, t_i \;:\; k_i \in Z \Big\},$$

where $t_1, \ldots, t_d \in R^n$ are linearly independent vectors, such that the translations τ_{t_i} (restricted to V) are automorphisms for each $i = 1, \ldots, n$. Also, the v_i's are non-equivalent by any translations, i.e., $K + v_i \bigcap K + v_j = \emptyset$ if $i \neq j$.

Next we need the following definitions.

(i) A free Z^d–action (by automorphism) on $G = (V, E)$ is a map $Z : Z^d \times V(G) \to V(G)$, verifying the properties (we consider the usual notation for an action $Z(k, v_x)$ for $k \in Z^d$, $v_x \in V$, namely, simply $Z(k, v_x) \equiv k\, v_x$):

(1) $k\,(l\,v_x) = (k+l)\,v_x, \forall\,k,l \in Z^d, \forall\,v_x \in V$.

(2) $0\,v_x = v_x, \forall\,v \in V$.

(3) $k\,v_x = l\,v_x \Rightarrow k = l, \forall\,k,l \in Z^d, \forall\,v_x \in V$.

(4) For any $k \in Z^d$, the map $Z^{(k)} : V \to V$, defined by $Z^{(k)}(v_x) = k\,v_x$, is an automorphism of G.

(ii) An orbit of a free action is the set

$$o_x = \{k\,v_x : k \in Z^d\} = \bigcup_{k \in Z^d} Z^{(k)}(v_x).$$

(iii) If $v_{x''} \in o_{x'}$, then $o_{x'} = o_{x''}$. For $Z^{(l)}(v_{x'}) = v_{x''}$, this readily follows from the identity

$$\bigcup_{k \in Z^d} Z^{(k)}(v_{x''}) = \bigcup_{k \in Z^d} Z^{(k+l)}(v_{x'}) = \bigcup_{k \in Z^d} Z^{(k)}(v_{x'}).$$

(iv) The set of all orbits under the action of Z^d is represented by V/Z^d, with $n = |V/Z^d|$.

Therefore, a *free-action graph* $G_f = (V_f, E_f)$ is an undirected, simple, connected and locally finite graph, equipped with a free Z^d-action by automorphism, with finitely many orbits.

Theorem A.1.1 *Let $L_\tau = (V_\tau, E_\tau)$ be a spatial lattice (with vertex set in R^n) as previously defined. Then, L_τ is also a free-action graph.*

Proof. Since the translations $\tau_{t_1}, \tau_{t_2}, \ldots, \tau_{t_n}$ are automorphisms of L_τ, we can define a free Z^d–action $Z : Z^d \times V_\tau \to V_\tau$ by automorphisms as follows

$$Z\big((k_1, k_2, \ldots, k_n), v_x\big) = v_x + k_1\,t_1 + k_2\,t_2 + \ldots + k_n\,t_n.$$

Note then that $Z^{(k)} = \tau_{k_1 t_1 + \ldots + k_n t_n}$. There are n orbits for this action: $o_i = K + v_i$, for all $i = 1, \ldots, n$. Hence, by definition L_τ is a free-action graph.

Theorem A.1.2 *Let $G_f = (V_f, E_f)$ be a free-action graph as previously defined. Then there exist a finite set S and an isomorphism $\phi : V(G) \to V_f$ such that $G = (Z^d \times S, E(G))$ is a formal periodic graph and $\big|S\big| = \big|V_f/Z^d\big|$.*

Proof. Let $S = \{1, 2, \ldots, n\}$ with $n = \big|V_f/Z^d\big|$. For each $s \in S$, let v_s be one element of the orbit o_s. We have that ($t \in S$)

$$v_s \notin o_t, \forall\,s \neq t. \tag{A.3}$$

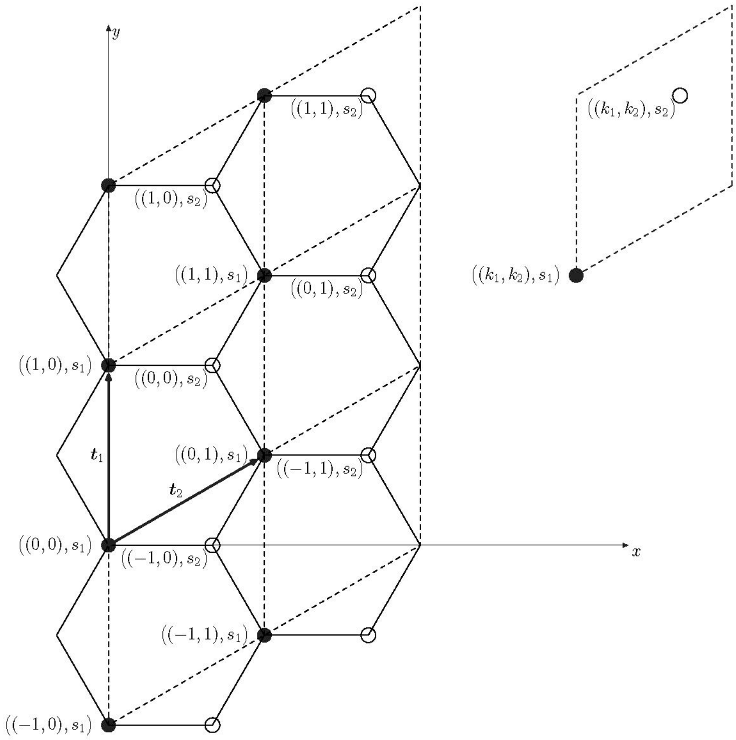


Figure 36 – The honeycomb (hexagonal) spatial lattice — whose unitary cell is shown in the right. The orbits $o_{(0,0)} = K$ and $o_{(1,0)} = K + (1,0)$ are represented, respectively, by filled and void circles. It is a periodic graph of vertex set $Z^2 \times \{s_1, s_2\}$ (with s_1, s_2 being 1 and 2).

By eq. (A.3) the sets $Z^{(k)}(S)$ have n elements for all $k \in Z^d$, so

$$V_f = \bigcup_{s=1}^{n} o_s = \bigcup_{k \in Z^d} Z^{(k)}(S).$$

Once the action is free this is a disjoint union. So we can define a bijection $\phi : Z^d \times S \to V_f$ by

$$\phi(k,s) = Z^{(k)}(s).$$

In this way, we can define the graph $G = (V(G), E(G))$, where the vertex set reads

$$V(G) = Z^d \times S,$$

whereas the edge set is naturally induced by

$$\big((k,s),(l,t)\big) \Leftrightarrow \big(\phi(k,s),\phi(l,t)\big).$$

Given the above definition for the edge set $E(G)$, it follows that the map ϕ is an isomorphism of graph. So the G_f and G are isomorphic. It remains to prove that G is a periodic graph, i.e., that A.1 holds true. And indeed

$$\begin{aligned} \big((k,s),(l,t)\big) \quad &\Leftrightarrow \big(Z^{(k)}(s), Z^{(l)}(t)\big) \Leftrightarrow \big(Z^{(0)}(s), Z^{(k-l)}(t)\big) \\ &\Leftrightarrow \big(\phi(0,s),\phi(k-l,t)\big) \Leftrightarrow \big((0,s),(k-l,t)\big). \end{aligned}$$

Finally

Theorem A.1.3 *Let $G_\tau = (V_\tau, E_\tau)$ be an arbitrary spatial lattice. Then G_τ is always isomorphic to some formal periodic G.*

Proof. This is a direct consequence of Theorems A.1.1 and A.1.2

We exemplify the previous construction addressing the honeycomb (hexagonal) lattice, see Fig. 36. Assume the two vectors in R^2

$$t_1 = (0, \sqrt{3}), \qquad t_2 = (3/2, \sqrt{3}/2).$$

We define the following set in R^2

$$K = \{k_1 t_1 + k_2 t_2 : k_1, k_2 \in Z\}.$$

The honeycomb spatial lattice L_{hon} can be defined as the simple graph with the vertex set V given as

$$V_{hon} = K \cup \big(K + (1,0)\big),$$

and the edge set E_{hon} defined as

$$E_{hon} = \Big\{(p, p+v_1), (p, p+v_2), (p, p+v_3) : p \in K\Big\},$$

for

$$v_1 = (1,0), \quad v_2 = (-1/2, \sqrt{3}/2), \quad v_3 = (-1/2, -\sqrt{3}/2).$$

It is clear that the translations τ_{t_1}, τ_{t_2} are automorphism of L_{hon}. We can define a free Z^2–action $Z : Z^2 \times V_{hon} \to V_{hon}$ by automorphisms through

$$Z\big((k_1, k_2), v_x\big) = v_x + k_1 t_1 + k_2 t_2.$$

The orbits of this action are

$$o_{(0,0)} = K, \qquad o_{(1,0)} = K + (1,0).$$

So, by the previous theorems, the honeycomb spatial lattice is a periodic graph, i.e., it is isomorphic to a periodic graph $G = (Z^2 \times \{s_1, s_2\}, E(G))$.

A.2 PROOF OF THEOREM 2.3.1 AND SOME SUPPORTING RESULTS FOR THE LEMMA 2.4.1.

We first recall the following well know result. Let $f : Z^d \to C$. The *discrete Fourier transform* (DFT) of f is the continuous function $\hat{f} : R^d \to C$, defined by

$$\hat{f}(w) = \sum_{k \in Z^d} f(k)\, \exp[+i\, w \cdot k].$$

If f has finite support, then for all $k \in Z^d$

$$f(k) = \frac{1}{(2\pi)^d} \int_B \hat{f}(w)\, \exp[-i\, w \cdot k]\, dw.$$

Next, we proof the Theorem 2.3.1. For simplicity we discuss $S = \{1, 2\}$ (for the general situation see below). We assume that the walk begins at $(0, t) \in Z^d \times S$. So, by definition $p_0((k, s), (0, t)) = \delta_{0k}\, \delta_{ts}$ (with t being 1 or 2).

The walk evolution is then governed by

$$p_{n+1}(k, s) = \sum_{(l,r) \in Z^d \times S} p((k, s), (l, r))\, p_n(l, r), \tag{A.4}$$

where we are using the notation $p_n(k, s) = p_n((k, s), (0, t))$. We define for all $n \in N$ the column vector

$$p_n(k) \quad = \quad \begin{pmatrix} p_n(k,1) \\ \\ p_n(k,2) \end{pmatrix}, \quad \text{in particular} \quad p_0(k) = \delta_{k0} \begin{pmatrix} 2-t \\ \\ t-1 \end{pmatrix}.$$

Further

$$p_1(k) \quad = \quad \begin{pmatrix} p_1(k,1) \\ \\ p_1(k,2) \end{pmatrix} = \begin{pmatrix} \Gamma_{11}(k) & \Gamma_{12}(k) \\ \\ \Gamma_{21}(k) & \Gamma_{22}(k) \end{pmatrix} \begin{pmatrix} 2-t \\ \\ t-1 \end{pmatrix}.$$

In this way, we can write Eq. (A.4) as

$$p_{n+1}(k) = \sum_{l \in Z^d} \Gamma(k-l)\, p_n(l).$$

Now, for $\hat{p}_{n+1}(w) = \sum_{k \in Z^d} p_{n+1}(k) \exp[i\, w \cdot k]$ we have that (taking into account the relation between Λ and Γ, Eq. (2.12))

$$\begin{aligned} \hat{p}_{n+1}(w) \quad &= \quad \sum_{k \in Z^d} \Big[\sum_{l \in Z^d} \Gamma(k-l)\, p_n(l) \exp[i\, w \cdot k] \Big] \\ &= \quad \sum_{l \in Z^d} \Big[\sum_{k \in Z^d} \Gamma(k-l)\, p_n(l)\, \exp[i\, w \cdot (k-l)] \\ &\qquad \times \exp[i\, w \cdot l] \Big] \Lambda(w)\, \hat{p}_n(w). \end{aligned} \tag{A.5}$$

Therefore ($e_1 = (2-t, t-1)^{\mathsf{T}}$)

$$\hat{p}_n(w) = \Lambda^n(w)\, e_1, \ \forall n \geq 0.$$

But p_n has finite support $\forall n$, thus

$$\begin{aligned} p_n(k) &= \frac{1}{(2\pi)^d} \int_B \exp[-i\, w \cdot k]\, \hat{p}_n(k)\, dw, \\ &= \frac{1}{(2\pi)^d} \int_B \exp[-i\, w \cdot k]\, \Lambda^n(w)\, (2-t, t-1)^{\mathsf{T}}\, dw. \end{aligned}$$

This leads to ($s = 1, 2$)

$$p_n(k, s) = p_n((k, s), (0, t)) = \frac{1}{(2\pi)^d} \int_B \exp[-i\, w \cdot k]\, [\Lambda^n]_{st}(w)\, dw. \qquad \text{(A.6)}$$

This equation is straightforwardly extended to the general case of $s, t \in S = \{1, \ldots, \mathcal{S}\}$, leading to the first relation in the Theorem 2.3.1.

The expression for the LGF (or probability generating function) follows from Eq. (A.6) with $S = \{1, \ldots, \mathcal{S}\}$ as ($-1 < z < +1$)

$$\begin{aligned} P((k, s), (0, t)) &= \sum_{n=0}^{\infty} p_n((k, s), (0, t))\, z^n \\ &= \frac{1}{(2\pi)^d} \int_B \sum_{n=0}^{\infty} \exp[-i\, w \cdot k]\, z^n\, [\Lambda^n]_{st}(w)\, dw, \\ &= \frac{1}{(2\pi)^d} \int_B \exp[-i\, w \cdot k] \big[\big(1 - z\, \Lambda(w)\big)^{-1}\big]_{st}\, dw. \end{aligned}$$

Above, the summation and integration can be exchanged due to uniform convergence (since $\rho(z\, \Lambda(w)) < 1$). Also, the last relation comes from the basic result $\sum_{n=0}^{\infty} A^n = (1 - A)^{-1}$, valid for $A \in C^{\mathcal{S} \times \mathcal{S}}$ with $\rho(A) < 1$.

To prove the Theorem A.2.1 next — necessary for the Lemma 2.4.1— we recall the following known identity. If $A \in C^{\mathcal{S} \times \mathcal{S}}$ is Hermitian ($A^\dagger = A$) and $\rho(A) < 1$, then

$$\ln\left[\mathrm{det}(1 - A)\right] = -\sum_{n=1}^{\infty} \frac{\mathrm{Tr}[A^n]}{n}, \qquad \text{(A.7)}$$

a direct consequence of $\ln[1 - x] = -\sum_{n=1}^{\infty} \frac{x^n}{n},\ x \in [-1, 1)$.

Theorem A.2.1 *If $A : X \subset R^d \to C^{m \times m}$ is a continuous function on a compact set X such that $A(w)$ is a Hermitian matrix and $\rho(A(w)) < 1\ \forall\, w \in X$, then*

$$\sum_{n=1}^{\infty} \frac{\mathit{Tr}[A^n(w)]}{n} = -\ln\left[\mathit{det}(1 - A(w))\right]$$

is uniformly convergent in X.

Proof. By Eq. (A.7)

$$\sum_{n=1}^{\infty} \frac{\text{Tr}[A^n(w)]}{n} = -\ln\left[\text{det}(1 - A(w))\right], \quad \forall w \in X.$$

Because $A(w)$ is Hermitian, the real function $w \in X \to \rho(A(w)) = |A(w)|_2$ is continuous. We define $0 \leq \eta < 1$ as the number

$$\eta = [w \in X]\max \rho(A(w)).$$

Then (for λ_i $(i = 1, 2, \ldots, m)$ eigenvalues of $A(w)$)

$$\left|\frac{\text{Tr}[A^n(w)]}{n}\right| = \left|\frac{1}{n}\sum_{i=1}^{m} \lambda_i^n(w)\right| \leq m\frac{\eta^n}{n}, \quad \forall\, w \in X.$$

By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} \frac{\text{Tr}[A^n(w)]}{n}$ converges uniformly on X.

A.3 PROOF OF THEOREM 2.6.1

Let $G = (V, E)$ be an undirected, simple, connected, and locally finite graph (of maximum degree $k_{max}(G)$) and $C_r(G)$ the set of all the closed walks of length r on G. We remark that $C_r(G)$ is a disjoint union of all the loops in $L_r(G)$ (remember that a loop is a equivalent class of closed curves, so a loop $l \in L(G)$ is a subset of $C_r(G)$). In this way

$$C_r(G) = \bigcup_{l \in L_r(G)} l.$$

Note that if G is a finite graph, $C_r(G)$ is a finite set. Because $|L_r(G)| \leq |C_r(G)|$ the sets $L_r(m)$ are also finite. Furthermore, leaving from v_x the number of ways which a walk can return to v_x after r steps cannot be greater than $(k_{max}(G))^r$. Since we have $|V(G)|$ vertices, then

$$\mathcal{C}_r(G) = |C_r(G)| \leq |V(G)|\,(k_{max}(G))^r. \tag{A.8}$$

Recalling that $|l| = 2\ell_l/m_l$, we have

$$\mathcal{C}_r(G) = \sum_{c \in C_r(G)} 1 = \sum_{l \in L_r(G)} \sum_{c \in l} 1 = \sum_{l \in L_r(G)} \frac{2r}{m_l}. \tag{A.9}$$

Theorem A.3.1 *Let $G = (V, E)$ be an undirected, simple, connected, and finite graph with maximum degree $k_{max}(G)$. For z in the the numerical interval $[0, 1/k_{max}(G))$, the RWLS partition function defined in section 2.6.1 can be written as*

$$\mathcal{Z}_G(z) = \exp\left[\sum_{r=1}^{\infty} J_r^G(z)\right], \tag{A.10}$$

where

$$J_r^G(z) = \sum_{l \in L_r(G)} w(l, z). \tag{A.11}$$

Proof. From Eqs. (2.42), (A.9) and (A.8), it reads (for notation simplicity we drop the superindex G in J_r^G)

$$\begin{aligned} J_r(z) &= \sum_{l \in L_r(G)} w(l, z) = \sum_{l \in L_r(G)} \frac{z^r}{m_l} = \frac{z^r}{2r} \sum_{l \in L_r(G)} \frac{2r}{m_l} \\ &= \frac{z^r\, \mathcal{C}_r(G)}{2r} \le |V(G)|\, (z\, k_{max}(G))^r. \end{aligned} \tag{A.12}$$

Given that $|V(G)|$ is finite and $z\, k_{max}(G) < 1$, the series $\sum_{r=1}^{\infty} J_r(z)$ is absolutely convergent. Thus, by the Cauchy theorem

$$\left(\sum_{r=1}^{\infty} J_r(z) \right)^m = \sum_{r=1}^{\infty} \left(\sum_{r_1 + \cdots + r_m = r} \prod_{i=1}^{m} J_{r_i}(z) \right), \tag{A.13}$$

where the term between $(\cdot)$ in the rhs of Eq. (A.13) is a finite series and always positive. So, in the following expression we can interchange the order of the summations as indicated

$$\begin{aligned} \exp\left[\sum_{r=1}^{\infty} J_r(z) \right] &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{r=1}^{\infty} J_r(z) \right)^m \\ &= 1 + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{r_1 + \cdots + r_m = r} \prod_{i=1}^{m} J_{r_i}(z). \end{aligned} \tag{A.14}$$

Now, from Eq. (A.11), i.e., $J_{r_i}(z) = \sum_{l \in L_{r_i}} w(l, z)$, Eq. (2.44) and the fact that over a Cartesian product, the product of sums is a sum of products, it reads

$$\begin{aligned} \sum_{r_1 + \cdots + r_m = r} \prod_{i=1}^{m} J_{r_i}(z) &= \sum_{r_1 + \cdots + r_m = r} \sum_{(l_1, \ldots, l_m) \in L(r_1, \ldots, r_m)} \prod_{i=1}^{m} w(l_i, z) \\ &= \sum_{(l_1, \ldots, l_m) \in L_r(m)} \prod_{i=1}^{m} w(l_i, z). \end{aligned} \tag{A.15}$$

Hence, considering Eqs. (A.14) and (A.15) we find

$$\begin{aligned} \exp\left[\sum_{r=1}^{\infty} J_r(z) \right] &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{r=1}^{\infty} \sum_{r_1 + \cdots + r_m = r} \prod_{i=1}^{m} J_{r_i}(z) \\ &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{r=1}^{\infty} \sum_{(l_1, \ldots, l_m) \in L_r(m)} \prod_{i=1}^{m} w(l_i, z). \end{aligned} \tag{A.16}$$

Lastly, the rhs of the second equality in Eq. (A.16) coincides with the rhs of Eq. (2.45). Thus, the Theorem A.3.1 follows.

Finally, we turn to the proof of Theorem 2.6.1. We consider $L = (V(L), E(L))$ an infinite q-regular lattice for the vertex set $V(L)$ being in $Z^d \times S$, with $|S| = \mathcal{S}$. We define for any $W \subset V(L)$

$$\begin{aligned} C_r^W(L) &= \{\{v_x, v_1, ..., v_{r-1}, v_x\} \in C_r(L) : v_x \in W\}, \\ \mathcal{C}_r^W(L) &= |C_r^W(L)|, \\ V^W(L) &= \{v_x \in V(L) : \exists\,(v_x, v_y) \in E(L) \;\&\; v_y \in W\}. \end{aligned}$$

If $W = \{v_x\}$, we simply write $C_r^W(L) = C_r^{\{v_x\}}(L) = C_r^{v_x}(L)$. If all the vertices forming a closed walk c are in W, $c \subset W$, otherwise $c \not\subset W$. Thus, we further define

$$C_r(G_n) = \{c \in C_r^{V_n}(L) : c \subset V_n\}. \tag{A.17}$$

For L, suppose a sequence of vertex-induced subgraphs $G_n(V_n, E_n)$ for which

$$V_n = \{(k, s) \in Z^d \times S : k \in [-n, n]^d\},$$

$|V_n| = (2n+1)^d \mathcal{S}$, $k_{max}(G_n) = q$ (so $\rho(A_{G_n}) \le q$), and $\lim_{n\to\infty} G_n = L$. For each $k \in Z^d$, the sets $U_k \subset Z^d \times S$ are (with $|U_k| = \mathcal{S}$)

$$U_k = \{(k, s) : s \in S\} = \{(k, 1), \ldots, (k, \mathcal{S})\}.$$

In special, $U_0 = \{(0, 1), \ldots, (0, \mathcal{S})\}$. Note that

$$V_n = \bigcup_{k \in [-n,n]^d \cap Z^d} U_k.$$

The boundary B_n of G_n is the set

$$B_n = \{v_x \in V_n : \exists\,(v_x, v_y) \in E(L) \;\&\; v_y \notin V_n\}.$$

Next we shall prove the following properties:

(i)

$$\lim_{n\to\infty} \frac{|B_n|}{|V_n|} = 0,$$

(ii)

$$\lim_{n\to\infty} \frac{J_r^{G_n}(z)}{|V_n|} = \frac{z^r}{2r} \lim_{n\to\infty} \frac{\mathcal{C}_r^{V_n}(L)}{|V_n|},$$

(iii)

$$\mathcal{C}_r^{V_n}(L) = \frac{|V_n|\, \mathcal{C}_r^{U_0}(L)}{\mathcal{S}}.$$

Proof of (i). For any $k \in Z^d$ the set $V^{U_k}(L)$ is finite once L is q-regular. Therefore, since L is periodic

$$\beta(k) = \beta = \max\{|l-k| : (l,s) \in V^{U_k}(L)\}$$

is independent on k and thus

$$\Big((k,s),(l,t)\Big) \in E(L) \Leftrightarrow |k-l| \leq \beta. \tag{A.18}$$

For $n = 1, 2, \ldots$, let $R_n = [-(n-\beta), n-\beta]^d \cap Z^d$. For all $n > \beta$, the β–interior $I_n^\beta \subset V(G_n)$ and β–boundary $B_n^\beta \subset V(G_n)$ are defined as

$$\begin{aligned} I_n^\beta &= \{(k,s) \in Z^d \times S : k \in R_n\} = \bigcup_{k \in R_n} U_k, \\ B_n^\beta &= V(G_n) - I_n^\beta. \end{aligned}$$

In Fig. 37 we illustrate these sets for the Archimedean lattice (4,8^2) considering $n = 2$. Given that $\beta \in N$ and $\left|R_n^d \cap Z^d\right| = (2n - 2\beta + 1)^d$, we have

$$\begin{aligned} |I_n^\beta| &= (2n - 2\beta + 1)^d \mathcal{S}, \\ |B_n^\beta| &= \big((2n+1)^d - (2n - 2\beta + 1)^d\big)\mathcal{S}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{|I_n^\beta|}{|V_n|} = \lim_{n\to\infty} \frac{(2n - 2\beta + 1)^d \, \mathcal{S}}{(2n+1)^d \mathcal{S}} = 1, \tag{A.19}$$

$$\lim_{n\to\infty} \frac{|B_n^\beta|}{|V_n|} = \lim_{n\to\infty} \frac{((2n+1)^d - (2n - 2\beta + 1)^d)\mathcal{S}}{(2n+1)^d \mathcal{S}} = 0. \tag{A.20}$$

For any $(k,s) \in I_n^\beta$ and $\big((k,s),(l,t)\big) \in E(L)$ we have by Eq. (A.18) that $|k-l| \leq \beta$ and so $(l,t) \in V_n$. In this way $(k,s) \notin B_n$ and $B_n \subset B_n^\beta$. Hence Eq. (A.20) leads to (i).

Proof of (ii). >From Eq. (A.17) we have the following disjoint decomposition $C_r^{V_n}(L) = C_r(G_n) + D_r(B_n)$, where $D_r(B_n) = \{c \in C_r^{V_n}(L) : c \not\subset V_n\}$. Thus, $\mathcal{C}_r^{V_n}(L) = \mathcal{C}_r(G_n) + |D_r(B_n)|$. But if $c \in D_r(B_n)$ then c visits a vertex in B_n and $|D_r(B_n)| \leq q^r \times |B_n| \times r$ since: there are at most q^r possibilities of a c of length r to be composed by a $v_x \in B_n$; in total there are $|B_n|$ vertices in the border; and v_x can be any of the r vertices along c.

Thence by (i) we have $\lim_{n\to\infty} |D_r(B_n)|/|V_n| = 0$ and

$$\frac{z^r}{2r} \lim_{n\to\infty} \frac{\mathcal{C}_r^{V_n}(L)}{|V_n|} = \frac{z^r}{2r} \lim_{n\to\infty} \frac{\mathcal{C}_r(G_n) + |D_r(B_n)|}{|V_n|} = \lim_{n\to\infty} \frac{J_r^{G_n}(z)}{|V_n|}, \tag{A.21}$$

proving (ii).

Proof of (iii). For all $s \in S$, $(0,s)$ and (k,s) are similar points in L and we have that $\mathcal{C}_r^{U_0}(L) = \mathcal{C}_r^{U_k}(L)$. Moreover

$$\mathcal{C}_r^{V_n}(L) = \sum_{k \in [-n,n]^d \cap Z^d} \mathcal{C}_r^{U_k}(L) = \frac{|V_n|}{\mathcal{S}} \mathcal{C}_r^{U_0}(L). \tag{A.22}$$

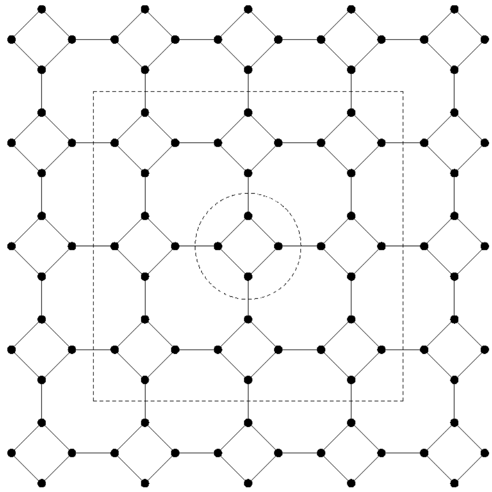


Figure 37 – The induced G_2 subgraph of the Archimedean lattice $(4,8^2)$, for which $\beta = 1$. $I_2^\beta \subset V(G_2)$ and $B_2^\beta \subset V(G_2)$ are, respectively, the β-interior and β-boundary sets of vertices (in the figure separated by the dashed square). The set U_0 is delimited by the dashed circle.

So (iii) holds true.

Lastly, since L is a periodic q–regular lattice then (see the discussion just after Eq. (2.4))

$$p_r(0,s) = p_r((0,s),(0,s)) = \frac{\mathcal{C}_r((0,s),(0,s))}{q^r},$$

and

$$\frac{z^r}{r}\mathcal{C}_r^{U_0}(L) = \frac{z^r}{r}\sum_{s=1}^{\mathcal{S}}\mathcal{C}_r((0,s),(0,s)) = \frac{z^r}{r}\sum_{s=1}^{\mathcal{S}} p_r(0,s)\, q^r. \tag{A.23}$$

By (ii), (iii) and Eq. (A.23), we have for $r = 1, 2, \ldots$

$$\lim_{n\to\infty}\frac{J_r^{G_n}(z)}{|V_n|} = \frac{1}{2\mathcal{S}}\frac{(z\,q)^r}{r}\sum_{s=1}^{\mathcal{S}} p_r(0,s).$$

In this way

$$\lim_{n\to\infty}\frac{J_r^{G_n}(z)}{|V_n|} = \frac{1}{2\mathcal{S}}\frac{(z\,q)^r}{r}\sum_{s=1}^{\mathcal{S}} p_r(0,s). \tag{A.24}$$

By Eq. (A.24), Theorem A.3.1 and Lemma 2.4.1, we conclude that for z in the interval $[0,1/q)$, in the thermodynamic limit (here meaning $n \to \infty$)

$$\begin{aligned} F_L(z) &= \lim_{n\to\infty}\tfrac{-\ln[\mathcal{Z}_{G_n}(z)]}{|V_n|} = -\lim_{n\to\infty}\tfrac{1}{|V_n|}\textstyle\sum_{r=1}^{\infty} J_r^{G_n}(z) \\ &= -\tfrac{1}{2\mathcal{S}}\textstyle\sum_{r=1}^{\infty}\left[\sum_{s=1}^{\mathcal{S}} p_r(0,s)\right]\tfrac{(z\,q)^r}{r} \\ &= \tfrac{1}{2\mathcal{S}}\tfrac{1}{(2\pi)^d}\textstyle\int_B \ln\left[\text{det}(1 - z\,q\,\Lambda(w))\right] dw, \end{aligned}$$

which is exactly the statement of Theorem 2.6.1.

APPENDIX B – PROOF OF SUPPORTING THEOREM AND LEMMAS FOR CHAPTER 3

B.1 2-PERIODIC GRAPHS.

Here we will define some notations and state important observations of a 2-periodic graph G as defined in **section II.C.**
We have two vectors v_1 and v_2, such that for each integer $n \geq 1$, and any $(k,l) \in Z^2$ we have an automorphism of G given by the translation

$$t_{(k,l)}(v) = v + kv_1 + lv_2,$$

note that

$$t^n_{(k,l)}(v) = v + n(kv_1 + lv_2)$$

If we define the vectors

$$v_{(k,l)} := kv_1 + lv_2,$$
$$v^n_{(k,l)} := n(kv_1 + lv_2),$$

we can write

$$t_{(k,l)}(x) = x + v_{(k,l)},$$
$$t^n_{(k,l)}(x) = x + v^n_{(k,l)},$$

Since G is 2-periodic we have that

$$V(G) \cap D = \{s_1, ..., s_{\mathcal{S}}\}.$$

so there is $a_1, ..., a_{\mathcal{S}}, b_1, ..., b_{\mathcal{S}} \in [0,1)$ such that

$$s_i = a_i v_1 + b_i v_2, \quad \forall i = 1, ..., \mathcal{S}.$$

Note that the sets $D(k,l)$ defined for all $(k,l) \in Z^2$ by

$$D(k,l) := \{t_{(k,l)}(av_1 + bv_2) : a, b \in [0,1)\}.$$

are disjoint and

$$R^2 = \bigcup_{(k,l)\in Z^2} D(k,l), \tag{B.1}$$

we also have that $D(0,0) = D$.

Since G is simple we have that $t_{(k,l)}$ have a a non-inversion property on the edges, meaning that if $e = \{v, w\} \in E(G)$ then form any $(k, l) \in Z^2$ we have

$$(t_{(k,l)}(v), t_{(k,l)}(w)) \neq (w, v)$$

We now also recall that for $n\,Z \oplus n\,Z$ (for a given $n \in N$) is the abelian additive group $\Lambda_n = (n\,Z \times n\,Z, +)$, where here the operation "+" is applied componentwise and the identity element is (0,0). Since Λ_n is a subgroup of Z^2, for each integer $n \geq 1$ we can define the quotient graph G_n as follows:

First Let $R_n \subset V(G) \times V(G)$ an equivalent relation on vertices given by: $(v_1, v_2) \in R_n$ if and only if it exists a $(k, l) \in Z^2$ such that

$$v_2 = t^n_{(k,l)}(v_1).$$

An equivalence class of the vertex v is the set

$$\Lambda_n v := \{t^n_{(k,l)}(v) : (k, l) \in Z^2\}$$

the set of all vertex classes is denoted by V/Λ_n.
We also have an equivalent relation $R'_n \subset E(G) \times E(G)$ on the edges defined as follows: $(e_1, e_2) \in R'_n$ if and only if it exists $(k, l) \in Z^2$ such that

$$\{t^n_{(k,l)}(v_1), t^n_{(k,l)}(w_1)\} = \{v_2, w_2\}$$

where $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$.
An equivalence class of the edge $e = \{v, w\}$ is the set

$$\Lambda_n e := \left\{\{t^n_{(k,l)}(v), t^n_{(k,l)}(w)\} : (k, l) \in Z^2\right\}$$

the set of all edge classes is denoted by E/Λ_n.
The *quotient graph* G/Λ_n is defined as follows

$$V(G/\Lambda_n) = V/\Lambda_n, \quad E(G/\Lambda_n) = E/\Lambda_n$$

and the incidence function ϕ_G is defined as

$$\phi_G(\Lambda_n e) = \{\Lambda_n v, \Lambda_n w\},$$

where $e = \{v, w\}$.
Note that $\Lambda_n e$ is a loop (link) if there exist an edge $\hat{e} \in \Lambda_n e$ such that its vertices are equivalent (not equivalent), in which case all the edges in the loop (link) $\Lambda_n e$ are equivalent (not equivalent).
Note that for $n = 1$, $\Lambda_1 = Z^2$ so we can write for any vertex $v \in V(G)$ and any $e \in E(G)$

$$Z^2 v = \Lambda_1 v, \quad Z^2 e = \Lambda_1 e$$

Lemma B.1.1 *Let B be a finite set with m elements, $T : B \to B$ a function such that*

a) T is injective,

b) $T(b) \neq b$, $\forall b \in B$,

c) $b \neq T(b') \Rightarrow T(b) \neq b'$, for all $b, b' \in B$.

Then m is even and there are $m/2$ elements $b_1, ..., b_{m/2} \in B$ such that

$$B = \{b_1, T(b_1), ..., b_{m/2}, T(b_{m/2})\}$$

Proof. The proof will use a finite number of steps.
Step 0: Let $b_1 \in B$,
Step 1: if $B - \{b_1, T(b_1)\} \neq \emptyset$, take

$$b_2 \in B - \{b_1, T(b_1)\},$$

we have $T(b_2) \notin \{b_1, T(b_1), b_2\}$
Step 2:if $B - \{b_1, T(b_1), b_2, T(b_2)\} \neq \emptyset$, take

$$b_3 \in B - \{b_1, T(b_1), b_2, T(b_2)\}$$

we have $T(b_3) \notin \{b_1, T(b_1), b_2, T(b_2), b_3\}$
This process must stop in a step p such that

$$B - \{b_1, T(b_1), ..., b_p, T(b_p)\} = \emptyset$$

So $m = 2p$ and

$$B = \{b_1, T(b_1), ..., b_{m/2}, T(b_{m/2})\}.$$

Theorem B.1.2 *Let $G = (V, E)$ be a 2-periodic, locally finite graph, then the fundamental quotient graph $G_1 := G/\Lambda_1$ is finite and locally finite, more precisely*

a) $V(G_1)$ have $\mathcal{S}$ elements and

$$V(G_1) = \{Z^2 s_i, i = 1, ..., \mathcal{S}\}$$

where

$$V(G) \cap D = \{s_1, ..., s_{\mathcal{S}}\}$$

b)

$$Deg\big[Z^2 s_i\big] = Deg[s_i], \quad \forall i = 1, ..., \mathcal{S}$$

c) $\mathcal{E} = E(G_1)$ *is finite and if* G *is also* $q-$*regular, then*

$$\mathcal{E} = \frac{q\mathcal{S}}{2}$$

Proof. To prove a) we need to prove two facts

i) $\bigcup_{i=1}^{\mathcal{S}} Z^2 s_i = V(G)$

ii) $\forall s, s' \in V(G) \cap D$ with $s \neq s'$

$$Z^2 s \neq Z^2 s'$$

Proof of i). Since it is clear that

$$\bigcup_{i=1}^{\mathcal{S}} Z^2 s_i \subset V(G),$$

we just need to prove the inclusion

$$V(G) \subset \bigcup_{i=1}^{\mathcal{S}} Z^2 s_i.$$

Let $v \in V(G)$ then by Eq. (B.1) there is $(k', l') \in Z^2$ and $a', b' \in [0, 1)$ such that

$$v = t_{(k',l')}(a'v_1 + b'v_2),$$

then

$$t_{(-k',-l')}(v) = a'v_1 + b'v_2 \in V(G) \cap D,$$

so for some integer $1 \leq \alpha \leq \mathcal{S}$, we have

$$t_{(-k,-l)}(v) = s_\alpha$$

then

$$v \in Z^2 s_\alpha \subset \bigcup_{i=1}^{\mathcal{S}} Z^2 s_i$$

So

$$V(G) \subset \bigcup_{i=1}^{\mathcal{S}} Z^2 s_i.$$

then

$$V(G) = \bigcup_{i=1}^{\mathcal{S}} Z^2 s_i.$$

So we proved i).

Proof of ii). Asume that $s \neq s$ and $Z^2 s = Z^2 s'$, so there is $a, b, a', b' \in [0, 1)$ and $(k, l) \in Z^2$ such that

$$\begin{aligned} s =& a v_1 + b v_2, \\ s' =& a' v_1 + b' v_2 \end{aligned}$$

and

$$av_1 + bv_2 = (a' + k)v_1 + (b' + l)v_2,$$

but the above equation implies that $(k, l) = 0$ so $s = s'$, which is a contradiction. So this prove ii), hence we proved a).

Now we will prove b).

Let $s \in D \cap V(G)$, since G is locally finite, we have

$$Deg(s) = q,$$

and since G is simple, we have q distinct vertices $r_1, ..., r_q$ and q links

$$\{s, r_1\}, ..., \{s, r_q\} \in E(G)$$

Consider the sets

$$\begin{aligned} A =& \{r_1, ..., r_q\}, \\ B =& \{r \in A : (s, r') \in R_1\} \\ C =& A - B = \{c_1, ..., c_{q-m}\} \end{aligned}$$

and let m be the number of elements in B.

Let $c, c' \in C$, if $\{s, c\}$ and $\{s, c'\}$ are equivalent then there is a $(k, l) \in Z^2$ such that

$$\begin{aligned} s + v_{(k,l)} &= s, \\ c + v_{(k,l)} &= c' \end{aligned}$$

or

$$\begin{aligned} s + v_{(k,l)} &= c', \\ c + v_{(k,l)} &= s \end{aligned}$$

The first pair of equation implies that $c = c'$ which is a contradiction and the second pair of equation implies that $c, c' \in B$ which is also a contradiction.

So we have $q - m$ different links in G_1 given by

$$Z^2\{s, c_1\}, ..., Z^2\{s, c_{q-m}\}$$

Now for all $b \in B$ exist a unique $(k_b, l_b) \in Z^2$ such that $(k_b, l_b) \neq 0$ and

$$t_{(k_b,l_b)}(b) = s$$

define the function $T : B \to B$ as

$$T(b) = t_{(k_b,l_b)}(s),$$

we will prove the following properties of T

i) T is injective,

ii) $T(b) \neq b$, $\forall b \in B,$

iii) $b = T(b') \Rightarrow T(b) = b'$, for all $b, b' \in B.$

Proof of i). If $T(b) = T(b')$, then

$$s + v_{(k_b,l_b)} = s + v_{(k'_b,l'_b)},$$

so $(k_b, l_b) = (k'_b, l'_b)$, now since

$$b + v_{(k_b,l_b)} = s = b' + v_{(k_b,l_b)},$$

then $b = b'$. So T is injective.
Proof of ii). We have

$$T(b) = s + v_{(k_b,l_b)}$$

and

$$s = b + v_{(k_b,l_b)}$$

So

$$T(b) = b + 2v_{(k_b,l_b)}$$

Now since $(k_b, l_b) \neq 0$ we have $T(b) \neq b$.
Proof of iii). Assume that $b = T(b')$ then

$$b = s + v_{(k_{b'},l_{b'})}$$

so

$$s = b + v_{(-k_{b'},-l_{b'})}$$

then $(-k_{b'}, -l_{b'}) = (k_b, l_b)$ Since

$$b' + v_{(k_{b'},l_{b'})} = s$$

then

$$b' + v_{(-k_b,-l_b)} = s$$

So

$$b' = s + v_{(k_b,l_b)} = T(b)$$

So iii) is Proved.
Now by Lemma B.1.1

$$B = \{b_1, T(b_1), ..., b_{m/2}, T(b_{m/2})\}$$

By the definition of B, for all $b \in B$ the equivalence classes of edges $Z^2\{s,b\}$ are loops in G_1, futhermore, since

$$\{s + v_{(k_b,l_b)}, b + v_{(k_b,l_b)}\} = \{T(b), b\}$$

we have that for all $b \in B$

$$\big(\{s,b\},\{s,T(b)\}\big) \in R_1,$$

in other words the edges$\{s,b\}$ and $\{s,T(b)\}$ are equivalent, so

$$Z^2\{s,b\} = Z^2\{s,T(b)\}$$

Next consider the edges

$$\{s,b_i\}, \quad \forall i = 1,...,m/2,$$

we will prove that for $i \neq j$

$$Z^2\{s,b_i\} \neq Z^2\{s,b_j\}.$$

If $\{s,b_i\}$ and $\{s,b_j\}$ are equivalent then there would be a $(k,l) \in Z^2$ such that

$$s + v_{(k,l)} = s,$$
$$b_i + v_{(k,l)} = b_j$$

or

$$s + v_{(k,l)} = b_j,$$
$$b_i + v_{(k,l)} = s$$

The first pair of equations above implies that $b_i = b_j$ which is a contradiction and the second pair implies that $(k,l) = (k_{b_i}, l_{b_i})$, and

$$T(b_i) = s + v_{(k,l)} = b_j$$

which is also a contradiction. So

$$Z^2\{s,b_i\} \neq Z^2\{s,b_j\}.$$

This means that G_1 have $m/2$ loops.
So

$$Deg[Z^2 s] = 2(m/2) + (q - m) = Deg[s],$$

so we proved b).
Proof of c). Using b) along with the Handshaking Lemma from Graph theory which states that for a locally finite graph $\hat{G}$ with finite set of vertices it holds that

$$\frac{E(\hat{G})}{2} = \sum_{v \in V(\hat{G})} Deg[v],$$

we conclude that $E(G_1)$ is finite, now if G is also $q-$regular then

$$\mathcal{E} = \frac{q\mathcal{S}}{2}.$$

Theorem B.1.3 *Let $G = (V, E)$ be a 2-periodic, locally finite graph, then for each integer $n \geq 1$, we have the relations*

a) $|V(G_n)| = n^2\mathcal{S}$,

b) $|E(G_n)| = n^2\mathcal{E}$.

Proof The case $n = 1$ was proved in Theorem B.1.2, now we will prove only b) because the proof of a) is analogous.
Assume $n, m \geq 2$ and define the following sets

$$\begin{aligned} I(m) =& \{0, ..., m-1\} \\ I(n,n) =& I(n) \times I(n) \\ I(m,n,n) =& I(m) \times I(n) \times I(n) \end{aligned}$$

Let $\dfrac{E}{Z^2} = \{Z^2 e_0, ..., Z^2 e_{m-1}\}$, where $m = \mathcal{E}$ and

$$e_i = \{v_i, w_i\} \in E(G), \quad \forall i = 0, ..., m-1.$$

Now define for all $i = 0, ..., m-1$ and for all $r, s = 0, ..., n-1$, the following edges in $E(G)$,

$$a_{irs} = \{v_i + v_{(r,s)}, w_i + v_{(r,s)}\}$$

We need to prove two facts:

i)
$$\bigcup_{(i,r,s)\in I(m,n,n)} \Lambda_n(a_{irs}) = E,$$

ii) $(i, r, s) \neq (i', r', s')$, then $\Lambda_n(a_{irs}) \neq \Lambda_n(a_{i'r's'})$,

Proof of i). It is clear that

$$\bigcup_{(r,s)\in I(n,n)} \Lambda_n(a_{irs}) = Z^2 e_i,$$

so

$$\bigcup_{(i,r,s)\in I(m,n,n)} \Lambda_n(a_{irs}) = \bigcup_{i\in I(m)} \Big(\bigcup_{(r,s)\in I(n)} \Lambda_n(a_{irs}) \Big)$$
$$= \bigcup_{i\in I(m)} Z^2 e_i = E$$

So i) is proved.
Proof of ii). Suppose $(i,r,s) \neq (i',r',s')$.
case (1). $i \neq i'$

If $\Lambda_n(a_{irs}) = \Lambda_n(a_{ir's'})$ then

$$a_{irs} \in \Lambda_n(a_{i'r's'}) \subset Z^2 e_i'.$$

Since $a_{irs} \in Z^2 e_i$, then $Z^2 e_i = Z^2 e_i'$ but this is a contradiction so $\Lambda_n(a_{irs}) \neq \Lambda_n(a_{i'r's'})$.
case (2). $i = i'$
If $\Lambda_n(a_{irs}) = \Lambda_n(a_{ir's'})$ then

$$a_{irs} \in \Lambda_n(a_{ir's'}).$$

So for some $(k,l) \in Z^d$

$$a_{irs} = \{v_i + v_{(r',s')} + v^n_{(k,l)}, w_i + v_{(r',s')} + v^n_{(k,l)}\}.$$

By the non-inversion property

$$v_i + v_{(r,s)} = v_i + v_{(r',s')} + v^n_{(k,l)},$$
$$w_i + v_{(r,s)} = w_i + v_{(r',s')} + v^n_{(k,l)}\}.$$

This implies that

$$(r-r', s-s') = (nk, nl)$$

Now since $(r,s), (r',s') \in I(n,n)$ then $|r-r'| < n$ and $|s-s'| < n$, so $(k,l) = 0$, then

$$(r,s) = (r',s'),$$

but this is also a contradiction, then in this case we also have that $\Lambda_n(a_{irs}) \neq \Lambda_n(a_{ir's'})$.
So ii) is proved, we conclude that

$$E(G_n) = \{\Lambda_n(a_{irs}) : (i,r,s) \in I(m,n,n)\}.$$

So $|E(G_n)| = n^2\mathcal{E} = n^2|E(G_1)|$.

Theorem B.1.4 *Let L be a 2-periodic, connected, locally finite graph, embedded in the plane such that each edge is a rectilinear segment and each face is a topological disc. Then*

$$\begin{aligned} f_L(K) &= \lim_{n\to\infty} \frac{\ln[Z_{G_n}(K)]}{|V(G_n)|} \\ &= \ln[2] + \frac{\mathcal{E}}{\mathcal{S}} \ln[\cosh(K)] \\ &\quad + \frac{1}{\mathcal{S}\, 8\, \pi^2} \int_B \ln\left[\mathit{det}(D_L(\tanh[K], w))\right]\, dw \end{aligned}$$

where

$$G_n = \frac{L}{\Lambda_n}.$$

Proof. By Theorem B.1.3, we have

$$\frac{|E(G_n)|}{|V(G_n)|} = \frac{|E(G_1)|}{|V(G_1)|} = \frac{\mathcal{E}}{\mathcal{S}}$$

and by formula (3.11)

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln[Z(G_n, K)]}{|V_n|} &= \ln[2] + \frac{\mathcal{E}}{\mathcal{S}} \ln[\cosh(K)] \\ &\quad + \lim_{n\to\infty} \frac{\ln[\mathcal{F}(G_n, \tanh[K])]}{|V_n|} \end{aligned}$$

So we have

$$\begin{aligned} f(K) &= \ln[2] + \frac{\mathcal{E}}{\mathcal{S}} \ln[\cosh(K)] \\ &\quad + \frac{1}{\mathcal{S}\, 8\, \pi^2} \int_B \ln\left[D_L(\tanh[K], w)\right]\, dw, \end{aligned}$$

. Next we will prove two theorems that are needed to prove Lemma 3.6.3.

Theorem B.1.5 *Let $J : R^2 \to R$ be a real analytic function such that for all $x, z \in (a,b)\times(c,d)$, $\partial_z(J(x,z)) \neq 0$ and let $(e,f) \subset (a,b)$, then*

a) If $\phi : (e,f) \to (c,d)$ is a function such that

$$J(x, \phi(x)) = 0, \quad \forall x \in (e,f),$$

then ϕ is real analytic.

b) If $J(x,c)J(x,d) < 0$ for all $x \in (e,f)$, there exists a real analytic function $\phi : (e,d) \to (c,d)$ such that

$$J(x, \phi(x)) = 0, \quad \forall x \in (e,f).$$

Proof.

Proof of a). For all $(x_0, z_0) \in (e, f) \times (c, d)$, by the real analytic Implicit Function Theorem ([123]), there exist a real analytic function

$$\phi_{x_0} : (x_0 - \delta, x_0 + \delta) \to (z_0 - \epsilon, z_0 + \epsilon)$$

such that

$$J(x, \phi_{x_0}(x)) = 0, \quad \forall x \in (x_0 - \delta, x_0 + \delta),$$

where $(x_0 - \delta, x_0 + \delta) \subset (e, f)$ and $(z_0 - \epsilon, z_0 + \epsilon) \subset (c, d)$.
Since $\partial_z(J(x, z)) \neq 0$, then

$$\phi_{x_0}(x) = \phi(x), \quad \forall x \in (x_0 - \delta, x_0 + \delta),$$

so ϕ is real analytic in (e, f).
Proof of b). Since $J(x, c)J(x, d) < 0$ for all $x \in (e, f)$, by the Intermediate Value Theorem, for each $x \in (e, f)$, there exist a number $z_x \in (c, d)$ such that $J(x, z_x)) = 0,$ By part a) the function $\phi : (e, f) \to (c, d)$, defined by $\phi(x) = z_x$ is real analytic.

Theorem B.1.6 *Let $J : R^2 \to R$ be a continuous function and $\epsilon > 0$.*

a) If r is the unique number in $[c, d]$ such that

$$J(a, r) = 0$$

and $\phi : (a, a + \epsilon) \to (c, d)$ is a function such that

$$J(x, \phi(x)) = 0, \quad \forall x \in (a, a + \epsilon).$$

then $\phi(a+) = r$

b) If s is the unique number in $[c, d]$ such that

$$J(b, s) = 0$$

and $\phi : (b - \epsilon, b) \to (c, d)$ is a function such that

$$J(x, \phi(x)) = 0, \quad \forall x \in (b - \epsilon, b).$$

then $\phi(b-) = s$

Proof. We only prove a) because the proof of b) is analogous.
Let (x_n) be a sequence in $(a, a + \epsilon)$ such that $x_n \to a$. We need to prove that $\phi(x_n) \to r$.

First we have that $\phi(x_n)$ is bounded so it have a limit point in $[c, d]$ (see [124]), so take any subsequence $\phi(x_{n_k})$ of $\phi(x_n)$ such that $\phi(x_{n_k}) \to l \in [c, d]$. Note that

$$(x_{n_k}, \phi(x_{n_k})) \to (a, l).$$

Since $J(x, z)$ is continuous and

$$J(x_{n_k}, \phi(x_{n_k})) = 0 \quad \forall k \geq 1.$$

we have that

$$J(x_{n_k}, \phi(x_{n_k})) \to J(a, l) = 0,$$

so by hypothesis $l = r$, hence the sequence $\phi(x_n)$ have a unique limit point, then

$$\phi(x_n) \to r$$

so $\phi(a+) = r$.